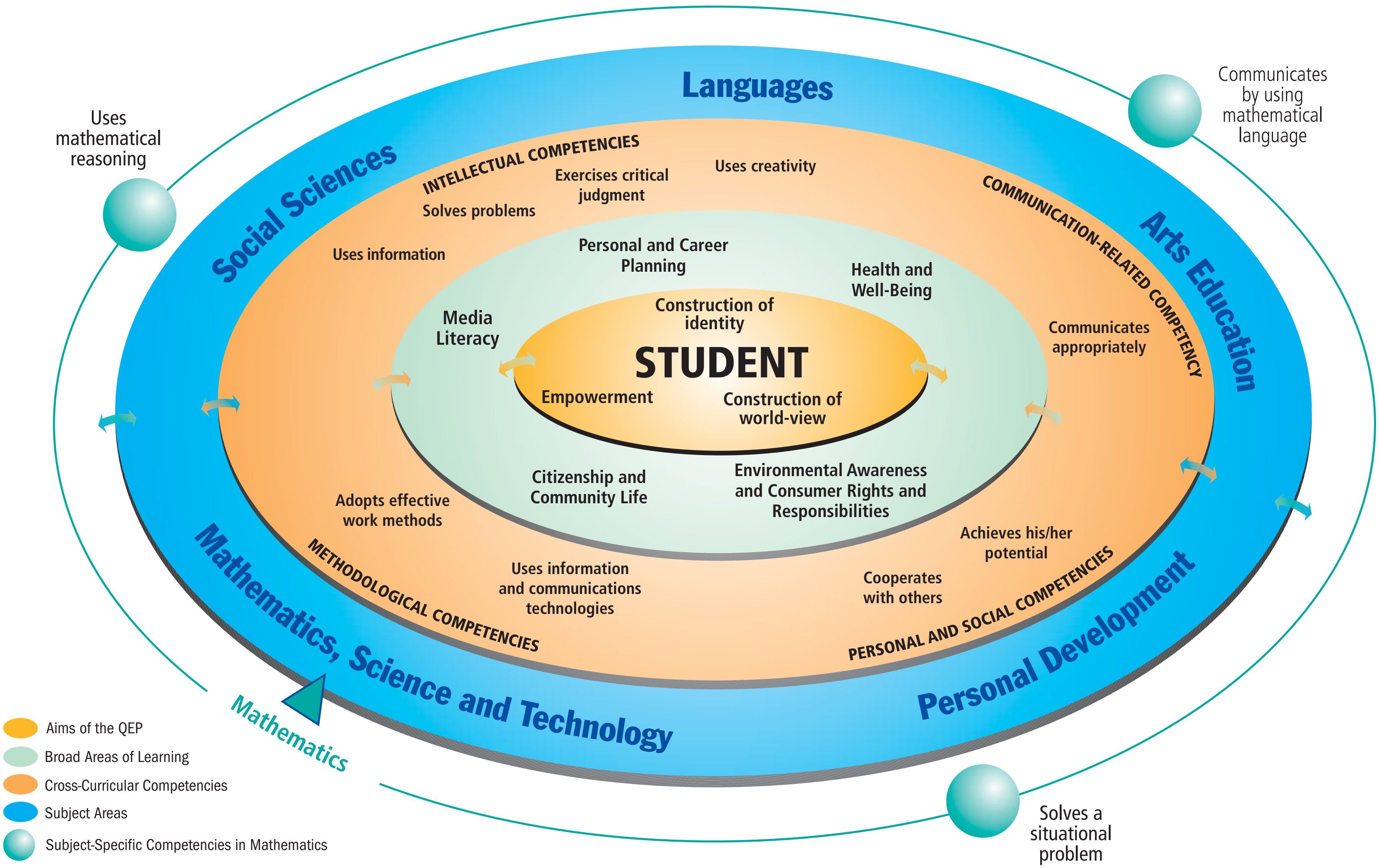
Mathematics

Making Connections: Mathematics and the Other Dimensions of the Québec Education Program (QEP)



Introduction to the Mathematics Program

Mathematics is a vast adventure in ideas; its history reflects some of the noblest thoughts of countless generations.

Dirk J. Struik

Mathematics is a science and a universal language that helps us understand reality. It makes an important contribution to a person's intellectual development, thereby shaping the construction of his or her identity. Its mastery is a major asset when carving out a place for oneself in a society that benefits from its many practical applications. It also remains essential for students who wish to continue their education in certain fields.

Mathematics is used in a multitude of everyday activities (e.g. the media, the arts, architecture, biology, engineering, computer science, the insurance industry, the design of various objects). Its many different applications cannot, however, be appreciated or understood without acquiring some basic knowledge of its various branches (arithmetic, algebra, statistics, probability, geometry). Because this knowledge makes students aware of the role mathematics plays in everyday life, it allows them to expand their world-view.

The many different situations that can be examined by means of mathematics or from which mathematics derives its structures show just how much it is related to other subject areas. With mathematics, we can interpret quantities by using arithmetic and algebra, space and shapes by using geometry, and random phenomena by using statistics and probability. As a result, mathematics is applied in a variety of areas (i.e. arts education, the social sciences, languages, personal and social development, and science and technology).

Since 1994, mathematics education in Québec has focused on getting students to solve problems, reason, establish connections and communicate. As was the case in the elementary-level Québec Education Program, these global objectives have been updated and consolidated in this program, which is centred on the development of three closely related competencies that are similar to those in the elementary curriculum:

- Solves a situational problem
- Uses mathematical reasoning
- Communicates by using mathematical language

The solving of situational problems is a central part of both mathematical and everyday activities and is examined from two perspectives. On the one hand, it is viewed as a process, which is embodied in the competency *Solves a situational problem.* On the other hand, problem solving is also an instructional tool that can be used in most mathematical learning processes. Moreover, it is of particular importance because the study of mathematical concepts requires the application of logical reasoning to situational problems.

The competency *Uses mathematical reasoning* is the cornerstone of all mathematical activity. In learning situations (situations involving applications, situational problems or other activities), students who use mathematical reasoning must organize their thinking by attempting to understand a body of knowledge and the interrelationships between these items of knowledge. In secondary school, they engage in three types of reasoning: analogical, inductive and deductive. They use analogical reasoning as they begin to recognize and learn from the similarities between topics in the different branches of mathematics, inductive reasoning when asked to derive rules or laws on the basis of their observations, and deductive reasoning when learning how to draw a conclusion on the basis of already accepted hypotheses and statements.

Developing the two above-mentioned competencies requires the use of a third competency (i.e. *Communicates by using mathematical language*). There are two objectives in developing this type of communication skill. The first is to become familiar with the elements of mathematical language (e.g. definitions, types of representation, symbols and notation), whereby students must also learn new words and new meanings for a known word. The second objective is to acquire the ability to formulate a message that involves explaining a procedure or a line of reasoning.

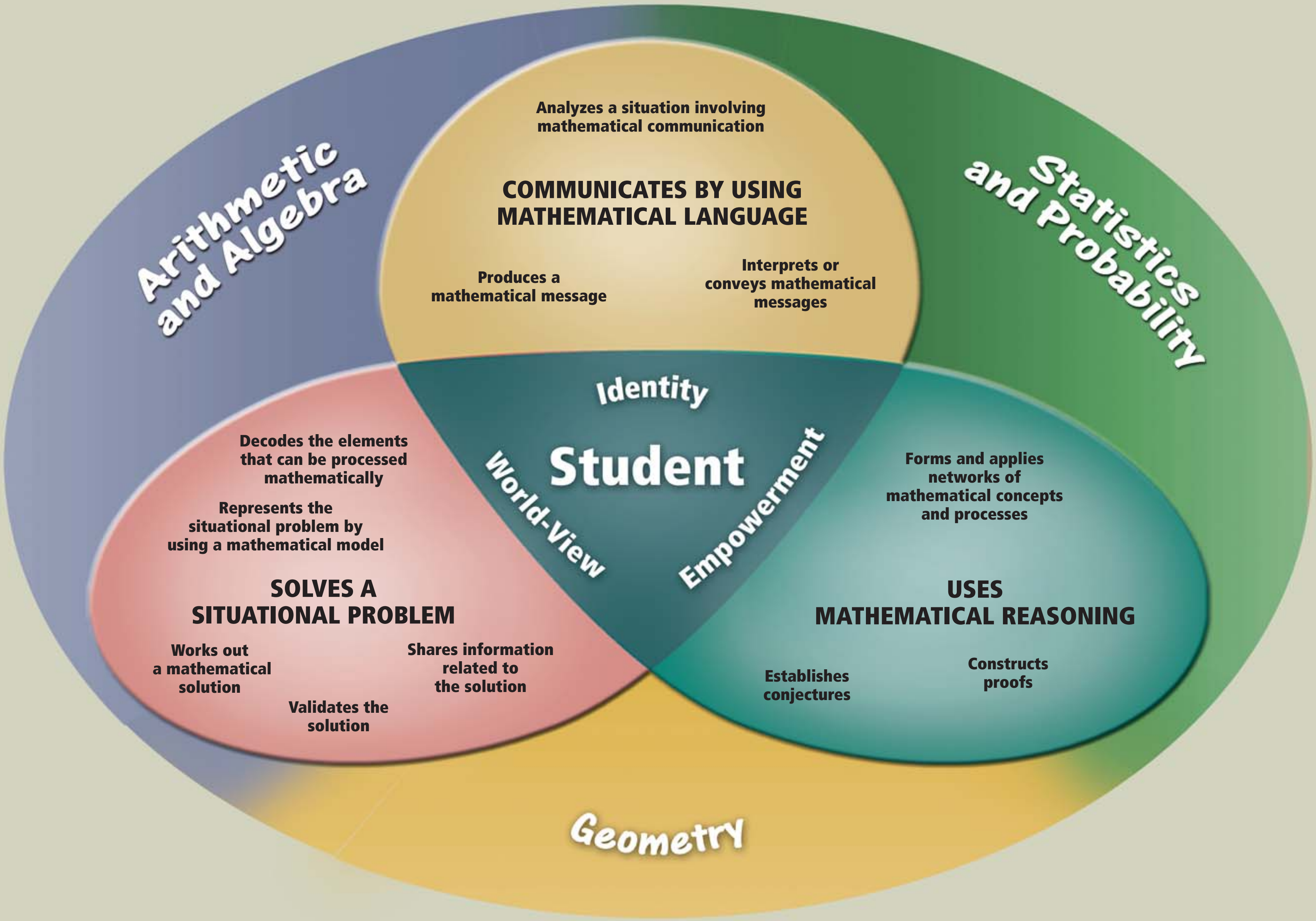
Although the three competencies developed in the program are, for all practical purposes, part and parcel of mathematical thinking, they are distinguished by the fact that they focus on different facets of that thinking. This distinction should make it easier to structure the pedagogical process without compartmentalizing the study of the elements specific to each competency. In addition, while mathematics, as a language and an abstraction tool, requires that the relationships between objects or elements of situations be examined in the abstract, secondary-level mathematics education is more effective when it involves real-world objects or situations.

Technology (e.g. calculator, computer) may be of considerable use in helping students deal with a given situation. By allowing students to explore, simulate and represent a greater number and variety of situations, technology fosters both the discovery and understanding of mathematical concepts and processes. It enables the students to carry out assigned tasks more efficiently and facilitates communication.

Furthermore, since the development of mathematics has been closely linked to human evolution, tracing the history of its development should be part of the program content. Such an approach should help students to understand the meaning and usefulness of mathematics and to discover how its historical evolution and the invention of certain instruments have been directly or indirectly related to the needs of different societies. This historical dimension should also help to illustrate that mathematical knowledge is the fruit of the extensive work of researchers with a passion for this subject, be they mathematicians, philosophers, physicists or artists.

The following diagram illustrates the interaction between the targeted competencies, the mathematical content and the student's overall development.

CONTRIBUTION OF THE MATHEMATICS PROGRAM TO THE STUDENT'S EDUCATION



Making Connections: The Mathematics Program and the Other Dimensions of the Québec Education Program

"Without the help of mathematics," the wise man continued, "the art could not advance and all the sciences would perish."
Júlio César de Mello e Souza alias Malba Tahan

Mathematics has a variety of everyday applications and is also connected to many components of the Québec Education Program. This connection is two-fold, meaning that mathematics education not only takes into account many of these components, but also contributes to them. For example, in examining themes pertaining to the broad areas of learning, students are asked to solve situational problems and use mathematical reasoning as well as the elements of mathematical language in order to clarify and explain different issues relating to their lives and concerns.

Connections With the Broad Areas of Learning

Through a variety of learning situations, students will also have the opportunity to make connections between, on the one hand, mathematical competencies and knowledge, and on the other, certain issues associated with the broad areas of learning or other subjects. The following are examples of these connections.

Personal and Career Planning

Students learn to carry out plans when solving situational problems. This ability contributes to their personal growth and helps them find their place in society. On a personal level, solving situational problems makes them aware of their identity and potential. They also gradually discover the role of mathematics in society by, for example, carrying out interdisciplinary projects involving related strategies and mathematical knowledge, while continuing to develop on a personal level.

Citizenship and Community Life

Students learn to take part in the democratic life of their school or classroom and to develop an attitude of openness to the world and respect for diversity. When making rules governing life in society, at school or in the classroom, students may use statistics, among other things. This would, for example, involve surveying and analyzing other people's opinion to improve their understanding of different problems and develop arguments with a view to making an informed decision.

Consumer Rights and Responsibilities

Students are encouraged to develop an active relationship with their environment, while maintaining a critical attitude toward consumer goods. Using their understanding of numbers and proportional reasoning, they interpret percentages, rates and indices in order to evaluate taxes, payment plans or discounts, for example. This gives them the opportunity to exercise their critical judgment and develop responsible strategies for consuming and using goods and services.

Media Literacy

Mathematical reasoning can contribute to the development of ethical and critical judgment, especially with regard to the media. By using different types of representations as well as proportional, probabilistic and statistical reasoning, students can make comparisons and gauge the difference between the reality of a situation and the way people perceive it. Conducting surveys helps students understand how survey results can form the basis for media articles and editorials.

Health and Well-Being

When encouraged to adopt a self-monitoring procedure in developing healthy lifestyle habits, students must interpret different messages. Their knowledge of statistics and probability can help them determine the relative importance of their lifestyle habits in promoting good health or the relative effectiveness of a course of treatment or a drug. They are often asked to present a summary of their work. This may also involve communicating by using mathematical language to analyze information and present it using different types of representations, which makes it easier to exercise critical judgment and to share information and points of view.

Environmental Awareness

Students are encouraged to develop an active relationship with their environment. By using their mathematical abilities pertaining to notation and representation (e.g. drawing plans, making scale drawings or constructing graphs to illustrate situations), they can demonstrate their understanding of environmental characteristics, the phenomena in the world around them or the interdependence of the environment and human activity.

Connections With the Cross-Curricular Competencies

When using their mathematical competencies, students are developing all the cross-curricular competencies. However, the cross-curricular competency *Solves problems* is a special case in that it involves many of the strategies associated with the mathematical competency *Solves a situational problem*. Although the key features of each are different, these competencies reflect a similar approach to asking questions and examining situations. As a result, their development leads to overlapping outcomes. In using their mathematical competencies, students also develop the cross-curricular competencies relating to the use of creativity, the processing of information, efficient work methods and effective communication.

Connections With the Other Subjects

Making connections between mathematics and other subjects enriches and contextualizes the learning situations in which the students will be developing their competencies. Conversely, some of the content of this program (e.g. the different types of representation, proportional reasoning, spatial sense and the processing of data) can be used in the study of other subjects.

There are many examples illustrating the multiplicity of links between mathematics and some of the other subject-specific competencies in the Québec Education Program. In studying science and technology, for instance, students who make the most of their scientific and technological knowledge also use mathematical reasoning and communicate through mathematical language when trying to explain phenomena by means of mathematical diagrams or models.

In the moral and religious instruction programs, students who take a reflective position on ethical issues may be required to use mathematical reasoning if they have to conduct a survey. They also communicate by using mathematical language when interpreting some of the information they are given.

Geometric reasoning and mathematical concepts and processes can prove to be very useful in creating individual and media images in the Visual Arts program, since this involves arranging two- and three-dimensional shapes in space.

When attempting to understand the organization of a territory in the Geography program, students use mathematical competencies and concepts to process statistical or other types of information, or to read and interpret maps or graphs. When students study history, mathematics can help them get a sense of the length of the period covered by a time line. Conversely, history can help students understand the evolution of important mathematical concepts.

Finally, it should be noted that the mastery of language and the different strategies related to the study of languages help students develop and use mathematical competencies.

The following diagram shows the links between mathematics and the other subjects.

SYSTEMIC VISION OF INTERDISCIPLINARY LINKS

MATHEMATICS

- **Solves a situational problem**
- **Uses mathematical reasoning**
- **Communicates by using mathematical language**

ARTS EDUCATION

Music

- Number sense for understanding the musical code

Drama

- Proportional reasoning
- Spatial sense

Visual Arts

- Geometric figures and transformations
- Spatial sense
- Proportional reasoning
- Cultural references

Dance

- Geometric figures and transformations
- Spatial sense

PERSONAL DEVELOPMENT

Physical Education and Health

- Number sense for interpreting information
- Processing and analyzing data
- Cultural references
- Spatial sense

Moral and Religious Education

- Number sense for interpreting information
- Processing and analyzing data
- Cultural references

SOCIAL SCIENCES

Geography, History and Citizenship Education

- Number sense for interpreting statistical data
- Proportional reasoning: scale of maps
- Processing data: asking questions; gathering, organizing, displaying, interpreting and analyzing
- Spatial sense: 2-D and 3-D representations, locating points on an axis and in a plane, geometric transformations and units of measure

LANGUAGE OF INSTRUCTION

- Uses language/talk to communicate and to learn
- Represents his/her literacy in different media
- Reads and listens to written, spoken and media texts
- Writes a variety of genres for personal and social use
- Reading and writing numbers, statements and conjectures
- Understanding a situational problem
- Using different types of representation
- Interpreting and organizing qualitative and quantitative data
- Making an argument, justifying and refuting

SCIENCE AND TECHNOLOGY

- Number and operation sense
- Proportional reasoning (e.g. plans, converting units of measure)
- Generalizing and constructing formulas
- Spatial sense (e.g. plans, technical objects)
- Processing data: asking questions; gathering, organizing, displaying, interpreting and analyzing data from experiments

Pedagogical Context

There is a real joy in doing mathematics, in learning ways of thinking that explain and organize and simplify. One can feel this joy discovering new mathematics . . . or finding a new way to explain . . . an old mathematical structure.
William P. Thurston

Learning and Evaluation Situations That Embrace Complexity

The three competencies in this program are interrelated and are developed in synergy, especially through learning situations that emphasize the students' active participation and a problem-solving approach and that also offer a certain measure of flexibility in choosing the types of representation to be used and in switching from one type of representation to another.

Students are active when they take part in activities involving reflection, manipulation, exploration, construction or simulation and have discussions that allow them to justify their choices, compare their results and draw conclusions. These situations require them to use their intuition, powers of observation, manual dexterity, and ability to listen and express themselves, which are of great help in acquiring concepts and processes and developing competencies.

To encourage the active participation of the students, the teacher must create an atmosphere that makes them feel at home in the class, which becomes a learning community. He or she devises a variety of activities and uses different pedagogical approaches, taking into account the needs, interests and prior learning of each student in order to help them develop their knowledge of mathematics.

It is also important that students be asked to work with situations that require justifications or answers to questions such as "Why?", "Is this always true?" or "What happens when . . .?". These should pertain to all branches of mathematics and force students to reason, acquire mathematical knowledge, interact and explain their procedure. In this way, they are encouraged to reflect on their actions and to deal with new situations.

The situational problems focus on obstacles to be overcome and about which the students formulate conjectures.[1] As a pedagogical tool, problem solving should be emphasized because it enables students to acquire a variety of invaluable concepts and skills. It applies to all the different branches of mathematics and allows students to use their creative and intellectual abilities. It is also conducive to the development of self-monitoring practices. By regularly using the approach involved in solving situational problems, students are able to:

– explore, devise, construct, broaden, expand, apply and integrate mathematical concepts and processes
– acquire the intellectual skills needed to develop a mathematical approach and way of thinking
– become aware of one's abilities and adopt an attitude of respect with regard to other people's point of view
– learn effective strategies

Exploration activities are extremely useful because they allow students to conjecture, simulate, experiment, develop arguments, build their knowledge and draw conclusions. For example, by analyzing different aspects of the relative positions of three lines in the same plane, students can identify several properties that can serve as the basis for validating other conjectures or solving certain situational problems.

Projects, or long-term activities that allow students to make connections with other subjects, are also good educational tools. The same is true of leisure activities, which usually stimulate the students' interest while helping them master a wide range of concepts and skills. Lastly, different communication situations, such as presentations, discussions and debates, are ideal opportunities for demonstrating the three competencies to be developed in this program.

All these activities can be carried out individually or in teams, in class or at home, depending on the development objectives involved and the pedagogical approaches used. They refer to real, fictitious, realistic, imaginary or purely mathematical situations, or practical situations that are relatively familiar to the students. They may be related to the other subjects, the students' environment, the broad areas of learning or the historical evolution of mathematics. Depending on the objective in question, activities may involve complete, superfluous, implicit or missing information. They may also lead to one or more outcomes, or they may lead nowhere.

1. In this program, the term conjecture refers to a statement that is thought to be true. The verb *to conjecture means to have a sense that a statement is true and to try to show that it is true.*

A Variety of Appropriate Materials

To exercise their competencies, students use different material resources depending on the activities involved (e.g. manipulatives and tools such as geometric blocks, objects, graph paper, a geometry set, a calculator and software). If necessary, they consult different sources of information in the library or on the Internet. They also call upon human resources, especially people in their school or community.

Although technology has an impact on mathematics and its applications, it cannot replace intellectual effort. However, it remains extremely useful. It helps students to learn about mathematics, explore more complex situations, manipulate large amounts of data, use a variety of representations, perform simulations and do tedious calculations more easily. As a result, they can focus on activities that are meaningful to them, develop their ability to do mental computation and study mathematical concepts and processes in greater detail. Dynamic geometry software is a good illustration of the value of technology. It allows students to consolidate their knowledge of geometry by making it possible to manipulate certain figures more easily, explore different situations, discover some of the properties of figures and construct them on the basis of their definitions and properties.

The different types of representations are essential for mastering concepts and are used in all the branches of mathematics, but switching from one type of representation to another helps students understand the situations they will encounter. For example, they will benefit from analyzing situations involving patterns or properties presented in different ways (e.g. in the form of verbal expressions, drawings, tables of values, graphs or symbolic expressions). However, in Secondary Cycle One, the translation from certain types of representations to others may not be covered at all or, if so, only occasionally. For instance, students in Secondary Cycle One are not expected to spend much time finding a rule given a graph or vice versa.

Appropriate Forms of Evaluation

To be consistent with the guiding principles of the program, evaluation, which is regarded as a learning tool, must focus on the degree to which the mathematical competencies have been developed as a whole. It should provide students with useful information on their learning progress, particularly regarding the extent to which they have mastered processes, subject-specific vocabulary, concepts and networks of concepts. It is important to continue evaluating program content, since students must have certain prerequisite knowledge if they are to expand networks of concepts and refine processes they will be using to develop the competencies. However, it is important to create situations that provide a reliable indication of the progress students have made in exercising their competencies, which involves bringing together enough of the elements of the program content. Various methods that take into account the students' different learning activities may be used (e.g. self-evaluation, an interview, an objective examination, an observation checklist, a logbook, a portfolio, or an oral or written presentation of a research project or solution).

The diagram on the next page illustrates a learning situation and shows how it can be connected to certain dimensions of the Québec Education Program.

OVERVIEW OF A SITUATION INCORPORATING DIMENSIONS OF THE QUÉBEC EDUCATION PROGRAM

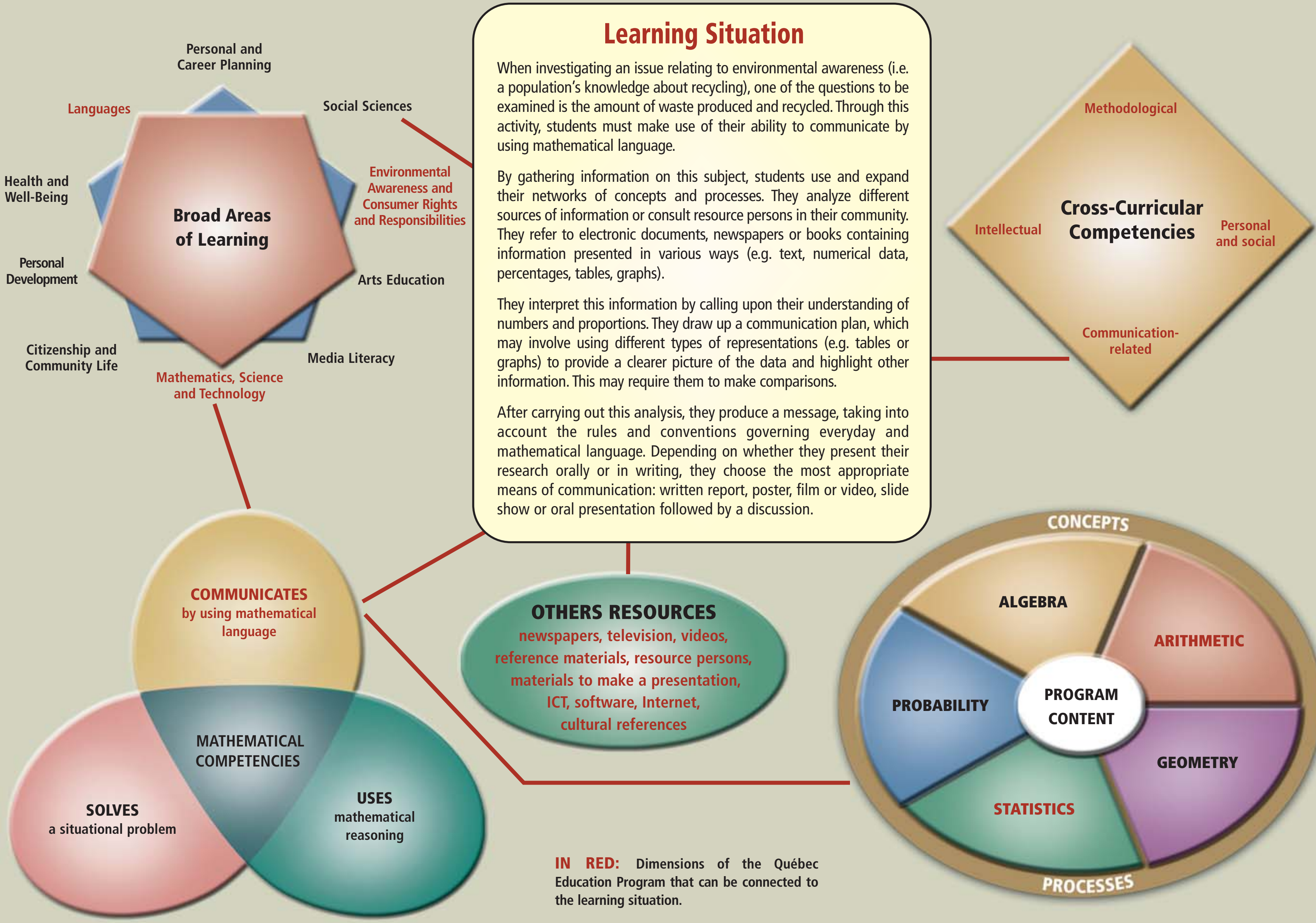
Learning Situation

When investigating an issue relating to environmental awareness (i.e. a population's knowledge about recycling), one of the questions to be examined is the amount of waste produced and recycled. Through this activity, students must make use of their ability to communicate by using mathematical language.

By gathering information on this subject, students use and expand their networks of concepts and processes. They analyze different sources of information or consult resource persons in their community. They refer to electronic documents, newspapers or books containing information presented in various ways (e.g. text, numerical data, percentages, tables, graphs).

They interpret this information by calling upon their understanding of numbers and proportions. They draw up a communication plan, which may involve using different types of representations (e.g. tables or graphs) to provide a clearer picture of the data and highlight other information. This may require them to make comparisons.

After carrying out this analysis, they produce a message, taking into account the rules and conventions governing everyday and mathematical language. Depending on whether they present their research orally or in writing, they choose the most appropriate means of communication: written report, poster, film or video, slide show or oral presentation followed by a discussion.



COMPETENCY 1 Solves a situational problem

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest, but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.
George Polya

Focus of the Competency

Solving a situational problem involves using a heuristic or discovery approach. In mathematics, this means being able to find a coherent solution to a situational problem under one of the following conditions:

- The situation has not been previously presented in the learning process.
- Finding a satisfactory solution involves using a new combination of rules or principles that the student may or may not have previously learned.
- The solution or the way in which it is to be presented has not been encountered before.

Solving a situational problem involves discernment, research and the development of strategies[2] entailing the mobilization of knowledge. It also requires the students to carry out a series of actions such as: decoding the elements that can be processed mathematically, representing the situational problem by using a mathematical model, working out a mathematical solution, validating this solution and sharing the information related to the situational problem and the proposed solution. This is a dynamic process that calls for the capacity to anticipate, backtrack and exercise critical judgment.

The ability to solve a situational problem is an effective intellectual tool that will help students to develop and improve other intellectual abilities that combine reasoning and creative intuition. This competency also makes it possible to use and continue developing the other two competencies in the program (i.e. *Uses mathematical reasoning* and *Communicates by using mathematical language*).

In elementary school, the students decoded situational problems that involved cases where information was missing or that had to be solved in several steps. They used various types of representations and strategies, which they developed in order to work out a solution. They learned to validate their solution and to explain it using mathematical language.

In Secondary Cycle One, the students continue developing this competency. They work with more complex situational problems that usually involve several branches of mathematics, depending on what is required. The following are examples of how each branch of mathematics contributes to the development of the competency.

- In arithmetic, the students use their number and operation sense as well as the relationships between these operations. They manipulate numerical expressions related to different sets of numbers, using processes for mental or written computation or using technology. They validate and interpret the numerical results in light of the context.
- In algebra, the students use different types of representations. They construct algebraic expressions, tables and graphs in order to generalize, interpret and solve the situational problem. They identify the unknown, solve equations to discover its value(s) and interpret them in light of the context.
- In probability theory, the students use tree diagrams, grids and networks to illustrate the situation, where possible, and to facilitate enumeration in simple combinatorial situations. They determine the sample space for a random experiment and calculate the probability of an event. They can then interpret this probability in light of the situation and make a decision related to that value, if necessary.
- In statistics, the students select appropriate tables and graphs in order to organize and analyze the data from a statistical report they have prepared or second-hand data whose source and context are known. If necessary, they use technological tools. If the students collect their own data, they do so by means of a questionnaire they have devised and, if applicable, they use different measures. In each case, they choose a representative sample.
- In geometry, the students make the transition from observation to reasoning. They state and use properties, definitions and relations to analyze and solve a situational problem. They construct figures, if necessary, using a geometry set or dynamic geometry software and manipulate numerical or algebraic expressions, especially to calculate lengths and areas. They interpret the numerical results and express them in units of measure appropriate to the situation.

2. See examples of these strategies on page 220.

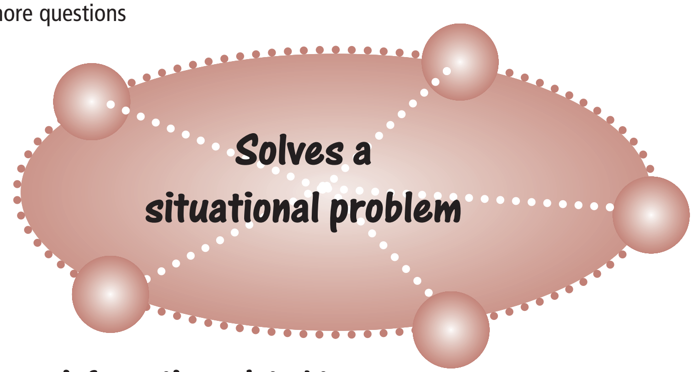
Key Features of Competency 1

Decodes the elements that can be processed mathematically

Derives information from various types of representations: linguistic, numerical, symbolic, graphic • If necessary, identifies any missing, additional or superfluous information • Identifies and describes the task to be performed by focusing on the question being asked or by formulating one or more questions

Represents the situational problem by using a mathematical model

Associates a suitable mathematical model with the situational problem • If necessary, compares the situational problem with similar problems solved previously • Recognizes similarities between different situational problems • Switches from one type of representation to another and formulates conjectures



Works out a mathematical solution

Uses appropriate strategies based on networks of concepts and processes • Describes the expected result by taking into account the type of information given in the problem • Estimates the order of magnitude of the result, if necessary • Organizes the information • Compares his/her work with the information given in the problem and the task to be performed

Shares information related to the solution

Provides a comprehensible and structured oral or written explanation of his/her solution • Takes into account the context, the elements of mathematical language and his/her audience

Validates the solution

Compares his/her result with the expected result • Rectifies his/her solution, if necessary • Assesses the appropriateness and efficiency of the strategies used by comparing own solution with those of his/her classmates and teacher or with those from other sources • Justifies the steps in his/her procedure

– Oral or written explanation showing that the student understands the situational problem

– Mobilization of mathematical knowledge appropriate to the situational problem

– Development of a solution (i.e. a procedure and a final answer) appropriate to the situational problem

End-of-Cycle Outcomes

By the end of Secondary Cycle One, students solve situational problems involving many items of given information and relating to one or more of the branches of mathematics. They use the various types of representations correctly, varying them from one situation to another depending on the context. They correctly use their relevant networks of mathematical concepts and processes. They work out a solution (a procedure and a final answer) by applying different strategies. They validate this solution and explain it using precise everyday and mathematical language.

Solving a situational problem involves using concepts and processes specific to each branch of mathematics.

- In arithmetic, students choose operations and apply the processes involved in performing them, taking into account the properties of these operations and the order in which they should be performed; they interpret the different types of numbers used, in light of the context.
- In algebra, students generalize a situation using an algebraic expression. When this expression is an equation, they determine and interpret the unknown in light of the context.
- In probability theory, students carry out activities involving enumeration and calculate probabilities. They interpret them and make decisions, where applicable.
- In statistics, students devise a questionnaire, if necessary, as well as organize, present and analyze survey data.
- In geometry, students construct figures, identify properties as well as the relationships between the properties of figures and use definitions. In calculating lengths and areas, they reason with regard to formulas by manipulating numerical or algebraic expressions and interpret their results.

COMPETENCY 2 Uses mathematical reasoning

We often hear that mathematics consists mainly of "proving theorems." Is a writer's job mainly that of "writing sentences?" A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration; and proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks.
Gian-Carlo Rota

Focus of the Competency

Using mathematical reasoning involves making conjectures and criticizing, justifying or refuting a proposition by applying an organized body of mathematical knowledge. This competency, which is essential for all mathematical activities, reflects a habit of mind that leads to a particular way of dealing with a situation. When students use mathematical reasoning, they determine how they will approach their work and organize their thinking. They follow the rules of inference and deduction and construct an organized and functional body of knowledge.

Mathematical reasoning and oral or written language are inextricably linked. Language (i.e. natural language, systems of representation, mathematical vocabulary and symbolism) is both a tool and object of reasoning. Language is also the vehicle of reasoning, since it can convey the conclusion that results from a line of reasoning and meet logical [3] or dialogical[4] criteria. Depending on whether you have to convince yourself, convince another student or someone unfamiliar with the given situation, or whether a solution must undergo a theoretical or practical validation, the thoroughness of your reasoning will vary as will the way in which it is presented. In using this competency, students also form and apply networks of mathematical concepts and processes, make conjectures, and validate them.

Reasoning plays a fundamental role in intellectual development, especially when it involves analyzing and dealing with various situations. It allows us to formulate a conjecture and to modify it if the given information or the learner's knowledge changes. When certain specific conditions are fulfilled (proof), reasoning may lead a person to change the truth value of a conjecture. Moreover, the use of mathematical systems of representation (e.g. diagrams, figures, graphs) as visual aids can lead to reasoning that is more intuitive, but no less rigorous. This type of reasoning, especially when it pertains to figures, must eventually give way to a more structured approach in which mathematical symbolism and the rules of proof will be used.

The development of this competency also calls for skills that are essential to the study of mathematics (e.g. expressing oneself and presenting an argument correctly, interpreting an everyday situation in mathematical terms, dealing with complex situations, working on a research project in teams or consulting textbooks and other school books on one's own).

The students began developing this competency in elementary school. They learned to build networks of mathematical concepts and processes by studying various patterns, making connections between numbers and operations, identifying geometric relationships, exploring activities involving chance and interpreting statistical data. They can mobilize these networks to solve the situational problems they are assigned and to justify actions and statements.

In Secondary Cycle One, students continue to build and use more extensive networks of concepts and processes consisting of elements of the program content associated with each branch of mathematics studied at this level. The following are examples of how each branch of mathematics contributes to the development of the competency.

- In arithmetic, students apply their understanding of numbers and operations when they use numbers written as decimals or fractions in comparing or estimating values, doing mental or written computation, and following the order of operations. They estimate the order of magnitude of a result, convert numbers written in fractional and decimal notation, apply divisibility criteria, and represent situations on a number line or in a Cartesian plane.

3. For example: if $a \times b = 0$, then, according to the multiplication property of zero, $a = 0$ or $b = 0$.
4. For example, argumentation.

Students use proportional reasoning when they observe that one quantity or magnitude is related to another by means of a given ratio. They use this type of reasoning to calculate a quotient, a rate (e.g. slope, speed, output) or an index; to perform operations on sequences of numbers or compare numbers belonging to those sequences; and to convert units or apply a percentage to a value. They also use proportional reasoning to construct and interpret tables, draw statistical graphs, analyze statistical or probability data and study similarity ratios in geometry.

- In algebra, students start to explore the meaning of algebraic expressions by, among other things, simplifying or multiplying them, solving equations with one unknown and modelling situations by describing them algebraically. They apply certain algebraic procedures to show that a conjecture is true, to solve equations or to apply formulas. They interpret algebraic expressions and associate them with different types of representations, thereby making it possible to coordinate the different elements of language.

- In probability theory, students learn to incorporate uncertainty into their reasoning by considering all the possibilities and including chance as a parameter. They study the relationships between two simple events (i.e. independence, equiprobability, complementarity or incompatibility). Using different diagrams, they derive combinatorial[5] rules and make connections based on the meaning and the properties of arithmetic operations. They can verify conjectures through experiments, simulations and the statistical analysis of the data they have collected.

- In statistics, students plan ways of collecting data, conduct surveys and apply reasoning to the data they have collected. They differentiate between the qualitative or quantitative aspects of the data. They use different types of reasoning to prepare a questionnaire and process the data they have collected, which involves organizing the data, choosing the most appropriate way of displaying it, interpreting it and formulating conclusions. The students exercise critical judgment when they evaluate the suitability of the quantitative and graphic methods used to process the data.

- In geometry, students use reasoning when they learn to recognize the characteristics of common figures, apply their properties and perform operations on plane figures by means of geometric transformations. They compare and calculate angles, lengths and areas, form nets for solids and draw them. They learn the definitions and properties of the figures they use to solve problems involving simple deductions. They determine unknown measures in different contexts.

The students make use of different types of reasoning (i.e. analogical, inductive and deductive), applying the reasoning appropriate to each branch of mathematics. Furthermore, as a prelude to the use of deductive reasoning, students should be introduced to certain basic rules of mathematical reasoning, such as those listed below:

– A mathematical statement is either true or false.
– Only one counterexample is required to show that a conjecture is false.
– One cannot conclude that a mathematical statement is true simply because several examples show it to be true.
– Observations or measurements based on a drawing do not prove that a conjecture is true, but may be used to formulate a conjecture.

When situations involving applications are used to evaluate this competency, students are required to apply a known combination of previously learned concepts and processes as well as certain aptitudes they have developed. These situations may be simple or complex. In a simple situation, evaluation focuses on whether the student has mastered the relevant network of concepts and processes. In a complex situation, it focuses on the mastery of more than one of these networks.

5. In this case, the term combinatorial refers to counting procedures.

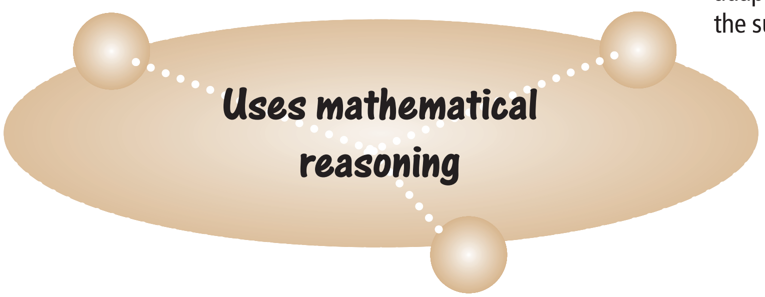
Key Features of Competency 2

Forms and applies networks of mathematical concepts and processes

Establishes organized and functional relationships between concepts and processes • Derives laws, rules and properties • Makes connections between different networks of concepts and processes • Uses different types of representations • Coordinates the elements of mathematical language pertaining to these networks

Establishes conjectures

Analyzes the conditions of a given situation • Organizes mathematical judgments • Forms a probable or plausible opinion • Becomes familiar with or formulates conjectures adapted to the situation • Evaluates the suitability of the stated conjectures



Constructs proofs

Chooses a type of representation • Uses methods associated with the type of representation selected • If necessary, uses counterexamples to clarify, adjust or refute conjectures • Organizes the results of his/her work • Repeats the exercise, if necessary

– Formulation of a conjecture appropriate to the situation[6]

– Correct use of the concepts and processes appropriate to the situation[7]

– Proper application of mathematical reasoning[8] suited to the situation

– Proper organization of the steps in an appropriate procedure

– Correct justification of the steps in an appropriate procedure

6. The situation referred to in this case is described in the last paragraph on page 201.

7. *Idem*.

8. In this case, the term *mathematical reasoning* means analogical, inductive or deductive reasoning and proportional, algebraic, geometric, arithmetic, probabilistic or statistical reasoning.

End-of-Cycle Outcomes

By the end of Secondary Cycle One, students use the different types of mathematical thinking to define the situation and propose conjectures. They apply concepts and processes appropriate to the situation and try different approaches in order to determine whether they should confirm or refute their conjectures. They validate them either by basing each step of their solution on concepts, processes, rules or statements that they express in an organized manner, or by supplying counterexamples.

Among other things, the use of mathematical reasoning involves applying concepts and processes relating to each branch of mathematics.

- In arithmetic, students call upon number and operation sense and use the equivalence between number representations or numerical expressions. They perform operations with numbers and apply the concepts of ratio, rate and proportion as well as multiplicative strategies, for example, in making conjectures related to proportional situations.
- In algebra, students interpret, construct and manipulate algebraic expressions.
- In probability theory, students use the concepts of enumeration and event to calculate probabilities.
- In statistics, students process data (i.e. they organize, display and analyze one or more aspects of a survey).
- In geometry, students make simple deductions based on definitions and properties in order to determine the value of unknown measurements, for example.

COMPETENCY 3 Communicates by using mathematical language

Such is the advantage of a well-constructed language that its simplified notation often becomes the source of profound theories.
Pierre-Simon de Laplace

Focus of the Competency

To communicate using mathematical language is to interpret and produce messages by combining everyday language with the specific elements of mathematical language (i.e. terms, symbols and notation). In certain situations, one can be more precise by using communication tools employed in mathematics. In developing this competency, the students will not only focus on the usual characteristics of an effective message (e.g. clarity and concision), but also become sensitive to the need for precision and rigour.

Producing or interpreting an oral or written message involving questions, explanations or statements related to mathematical activities forces students to clarify their thoughts and provides them with the opportunity to learn about mathematical concepts and processes and to reinforce that knowledge. In using this competency, students are also required to analyze a mathematical communication situation and to produce, interpret or convey mathematical messages.

The communication process benefits all those who take part in discussions, if only because the circulation of information is mutually rewarding. It is especially useful to the person conveying the message because the need to explain our understanding of a mathematical situation or concept helps us improve and deepen that understanding.

In addition to developing this competency, it is also important to master the language specific to mathematics itself, which is abstract in some respects. For example, a circle or an equation does not exist as such in nature. The students must become familiar with the elements of mathematical language, namely terms, symbols and notation, and learn to choose types of representation (i.e. numerical, symbolic, graphic, linguistic) that suit various situations. They must be able to use these different types of representations and be able to switch from one to another with ease. Several definitions of terms require special attention, since they become more detailed as students' learning progresses. For example, the definition of a square in Elementary Cycle One is usually less complex and detailed than the one used by students in Secondary Cycle One.

In elementary school, students interpreted and produced oral or written messages, using different types of representation. They refined their choice of mathematical terms and symbols. They compared information from various sources. In discussions with classmates, they analyzed different points of view and adjusted their message if necessary.

In Secondary Cycle One, the elements of mathematical language brought into play to develop and use this competency are part of the program content that relates to each branch of mathematics. The following are examples of how each branch of mathematics contributes to the development of the competency.

- In arithmetic and algebra, the students communicate when producing and interpreting symbolic expressions used to generalize and model relationships between numbers.
- In statistics and probability theory, the students communicate when counting data values, and organizing, analyzing and interpreting data.
- In geometry, the students communicate when describing and interpreting a figure in order, for example, to reproduce it. When looking for unknown measurements, they use units of measure and produce or interpret formulas.

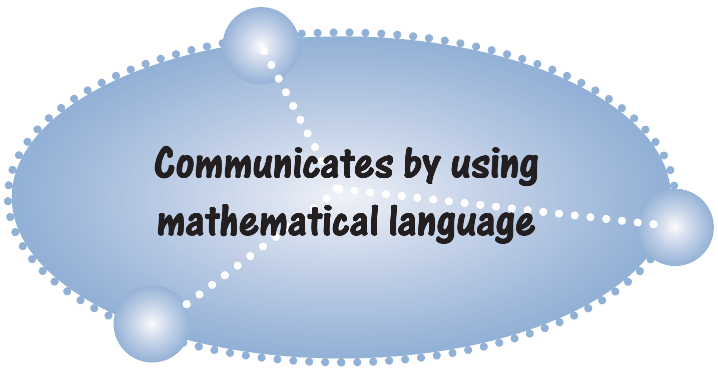
In all cases, the students are required to communicate when they make conjectures based on networks of mathematical concepts and processes, since they must present their arguments and decisions and justify their solution.

Students develop this competency when using the other two subject-specific competencies, since it is closely related to the conceptualization and explanation of the knowledge, processes and procedures applied in using mathematical reasoning or solving situational problems.

Key Features of Competency 3

Analyzes a situation involving mathematical communication

Identifies the purpose of the message • Distinguishes between the everyday and mathematical meaning of various terms • Consults different sources of information, when necessary • Organizes his/her ideas and establishes a communication plan



Interprets or conveys mathematical messages

Expresses his/her ideas using mathematical language, taking into account its rules and conventions as well as the context • Validates a message to make it more comprehensible, if necessary • Summarizes information • Has discussions based on mathematical messages

Produces a mathematical message

Chooses the elements of mathematical language that suit the context and the message • Associates images, objects or concepts with mathematical terms and symbols, depending on the context • Selects types of representations that suit the message and the audience

- Correct interpretation of a message involving at least one type of mathematical representation suited to the situation
- Production of a message suited to the context, using appropriate mathematical terminology and following mathematical rules and conventions

End-of-Cycle Outcomes

By the end of Secondary Cycle One, students interpret or produce oral or written messages relating to all the branches of mathematics covered in this program. They use appropriate mathematical and everyday language and choose different types of suitable representations. The messages are clear and coherent given the situation and the audience. If necessary, students can explain them.

Different concepts and processes must be brought into play in each branch of mathematics.

- In arithmetic and algebra, students use symbolic expressions that result from modelling or generalizing the relationships between numbers.
- In statistics and probability theory, students explain the counting procedures they use, and they organize, represent and interpret data.
- In geometry, students describe and interpret geometric figures. They produce and interpret formulas to find unknown measurements.

Program Content

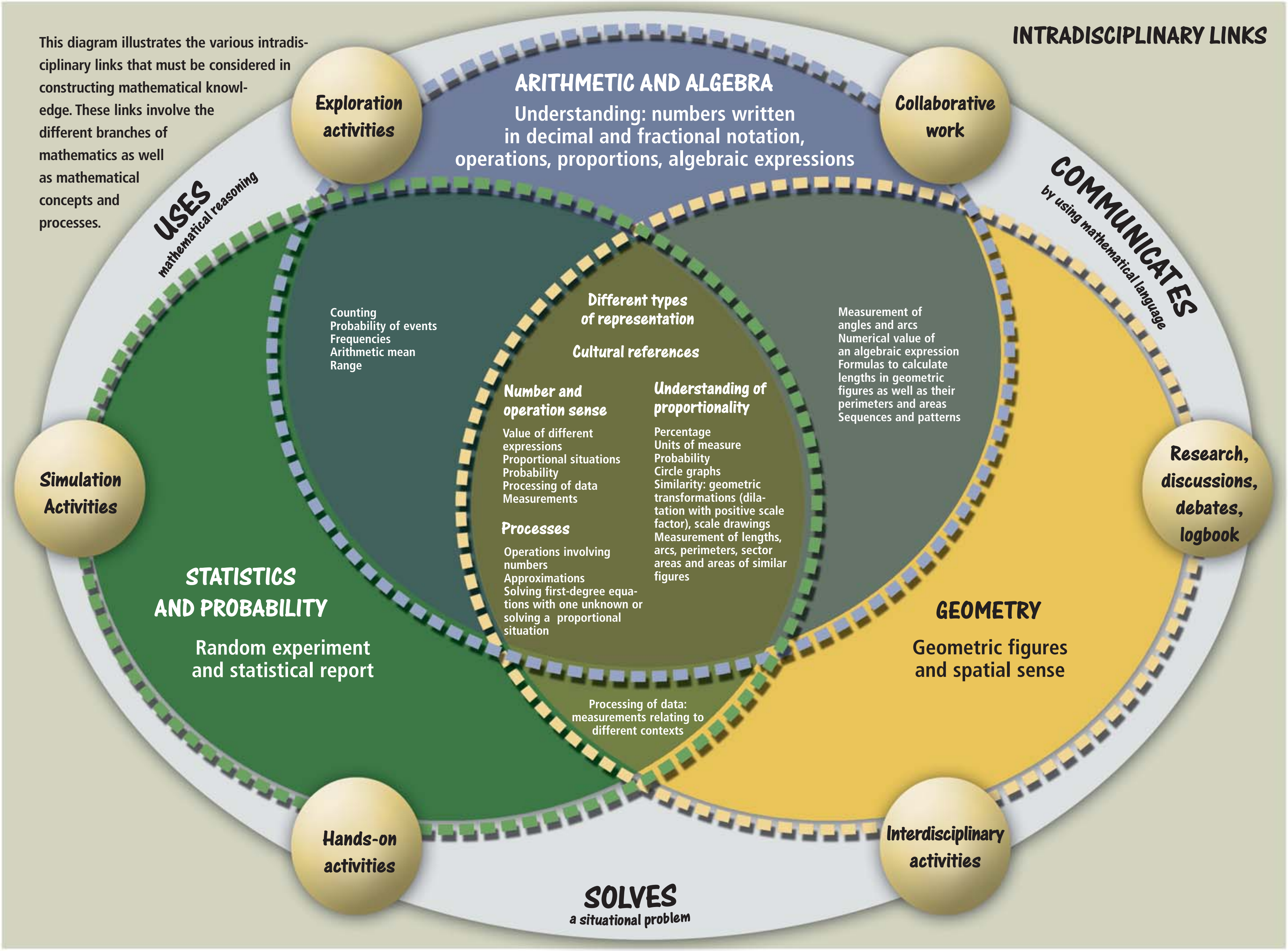
Time was when all the parts of the subject were dissevered, when algebra, geometry, and arithmetic either lived apart or kept up cold relations of acquaintance confined to occasional calls upon one another; but that is now at an end; they are drawn together and are constantly becoming more and more intimately related and connected by a thousand fresh ties, and we may confidently look forward to a time when they shall form but one body with one soul.

James Joseph Sylvester

The competencies and the mathematics program content are closely related. The students' ability to apply concepts and processes to situational problems or situations involving applications indicates the extent to which the students have mastered these concepts and processes. This ability is therefore an important factor in the development of the first two competencies (i.e. *Solves a situational problem* and *Uses mathematical reasoning*). To develop the competency *Communicates by using mathematical language*, the students refer to their existing knowledge of terminology and symbolism relating to mathematical concepts and acquire new knowledge in this regard. This is another example of how a competency is inextricably linked to the learning content.

This section outlines mathematical concepts and processes, learning processes and cultural references pertaining to arithmetic, algebra, probability theory, statistics and geometry. It ends with examples of strategies used to solve situational problems. Since the prime objective of the Québec Education Program is to develop competencies, most of the concepts and processes must be constructed by the students and reapplied in a variety of contexts. On the one hand, the program content can be viewed in a linear fashion, since the mathematical edifice is constantly built on prerequisites. On the other hand, the program content can be used to highlight the links between the different branches of mathematics and with the other subjects. However, the learning content should be examined symbiotically because the branches of mathematics are interconnected in such a way that the principles associated with one branch can be useful in understanding the principles pertaining to another branch. For its part, mathematical language consists of terms, notation, symbols and different types of representations that must be mastered to ensure clear and unambiguous communication.

The sections on cultural references provide suggestions that will help students to situate mathematical concepts in a social and historical context, to study their evolution and to identify the problems that gave rise to the development of certain processes as well as the needs these concepts fulfilled. These references should enable students to appreciate how mathematics influences their everyday lives and how mathematicians have contributed to its development. Be it through the use of situational problems, historical vignettes, research projects, interdisciplinary activities or a logbook, it is important to devise learning situations that allow students to discover the different roles played by mathematics as well as the various aspects of its history. This will enable them to make connections with the other subject areas and to develop an informed, aesthetic or critical view of the world.


INTRADISCIPLINARY LINKS
This diagram illustrates the various intradisciplinary links that must be considered in constructing mathematical knowledge. These links involve the different branches of mathematics as well as mathematical concepts and processes.
Exploration activities
Collaborative work
Simulation Activities
Research, discussions, debates, logbook
Hands-on activities
Interdisciplinary activities
USES mathematical reasoning
COMMUNICATES by using mathematical language
SOLVES a situational problem
ARITHMETIC AND ALGEBRA
Understanding: numbers written in decimal and fractional notation, operations, proportions, algebraic expressions
STATISTICS AND PROBABILITY
Random experiment and statistical report
GEOMETRY
Geometric figures and spatial sense
Counting
Probability of events
Frequencies
Arithmetic mean
Range
Measurement of angles and arcs
Numerical value of an algebraic expression
Formulas to calculate lengths in geometric figures as well as their perimeters and areas
Sequences and patterns
Different types of representation
Cultural references
Number and operation sense
Value of different expressions
Proportional situations
Probability
Processing of data
Measurements
Processes
Operations involving numbers
Approximations
Solving first-degree equations with one unknown or solving a proportional situation
Understanding of proportionality
Percentage
Units of measure
Probability
Circle graphs
Similarity: geometric transformations (dilatation with positive scale factor), scale drawings
Measurement of lengths, arcs, perimeters, sector areas and areas of similar figures
Processing of data: measurements relating to different contexts

Arithmetic[9]

Numbers are intellectual witnesses that belong only to mankind.
Honoré de Balzac

In elementary school, students developed an understanding of numbers and operations involving natural numbers, fractions and decimals. They are able to convert numerals from fractional notation to decimal notation or to a percentage. They identified the properties of operations as well as the relationships between them. They know how to follow the order of operations in simple sequences of operations. They were introduced to the concept of integers. They are able to perform operations mentally or in writing with natural numbers and decimals.[10] Finally, they used objects and diagrams to perform certain operations[11] involving fractions.

In Secondary Cycle One, they develop and master the following concepts and processes:

Concepts

Number Sense With Regard to Decimal and Fractional Notation and Operation Sense

– Reading, writing, various representations, patterns, properties
– Fractional, decimal and exponential (integral exponent) notation; percentage, square root
– Properties of divisibility (by 2, 3, 4, 5, 10)
– Rules of signs for numbers written in decimal notation
– Equality relation: meaning, properties and rules for transforming numerical equalities (balancing equalities)
– Inverse operations: addition and subtraction, multiplication and division, square and square root
– Properties of operations:
  • Commutative and associative properties
  • Distributive property of multiplication over addition or subtraction and factoring out the common factor
– Order of operations and the use of no more than two levels of parentheses in different contexts

Processes

Different Ways of Writing and Representing Numbers

– Estimating the order of magnitude
– Comparing
– Using a variety of representations (e.g. numerical, graphic)
– Recognizing and using equivalent ways of writing numbers:
  • Decomposition of numbers (e.g. additive, multiplicative)
  • Equivalent fractions
  • Simplification and reduction
– Switching from one way of writing numbers to another or from one type of representation to another
– Transforming arithmetic equalities
– Locating numbers on a number line, abscissa (x-coordinate) of a point

Note
Positive or negative numbers written in decimal or fractional notation are used on a number line or in a Cartesian plane. In switching from one way of writing numbers to another, the students should work with positive numbers.

9. Proportions are studied after arithmetic.
10. There are certain restrictions relating to the magnitude of the natural numbers and decimals that can be used. In this regard, refer to the elementary-level Québec Education Program.
11. For the addition and subtraction of fractions, the denominator of one fraction must be a multiple of the denominator of the other fraction. Fractions are multiplied by natural numbers only. The multiplication and division of fractions is not covered at the elementary level.

Concepts

Note

This program focuses on positive and negative rational numbers written in decimal or fractional notation. Sets of numbers are not studied systematically in this cycle, but students should still be encouraged to use the proper terms learned in elementary school (natural numbers, integers, decimals).

Learning activities must focus on helping students develop an understanding of numbers, operations and the concept of equality.

Depending on the context or the needs involved, students may also use other properties of divisibility (e.g. by 6, 9, 12 or 25).

Knowledge of the properties of operations helps students think of equivalent ways of writing numbers and operations, which simplifies computations and can eliminate dependence on a calculator.

Knowledge of the order of operations helps students understand and appreciate the efficiency of technology.

Processes

Operations Involving Numbers Written in Decimal and Fractional Notation

- Estimating and rounding numbers in different situations
- Looking for equivalent expressions
- Approximating the result of an operation
- Simplifying the terms of an operation
- Mental computation: the four operations, especially with numbers written in decimal notation, using equivalent ways of writing numbers and the properties of operations
- Written computation: the four operations involving numbers that are easy to work with (including large numbers) and sequences of simple operations performed in the proper order (numbers written in decimal notation), using equivalent ways of writing numbers and the properties of operations

Examples (for Mental or Written Computation)

$$15 \times 102 = 15(100 + 2) = 15 \times 100 + 15 \times 2 = 1\,500 + 30 = 1\,530$$

$$2\frac{1}{4} \times 3\frac{1}{2} = 2 \times 3 + 2 \times \frac{1}{2} + \frac{1}{4} \times 3 + \frac{1}{4} \times \frac{1}{2} = 6 + 1 + \frac{3}{4} + \frac{1}{8} = 7\frac{7}{8}$$

$$3.5 \times 6 - 3.5 \times 4 = 3.5\,(6 - 4) = 7$$

- Use of a calculator: operations and sequences of operations performed in the proper order

Note

In these operations, the only negative numbers that should be used are those written in decimal notation.

Students use a technological tool for operations in which the divisors or multipliers have more than two digits.

For written computation, the understanding and mastery of processes is more important than the ability to do complex computations.

Students will learn to use technology when appropriate.

Learning Process

Expressing situations mathematically, anticipating the numerical results of operations and interpreting numerical results in light of the context will help students develop an understanding of numbers and operations.

If necessary, students visualize operations by using concrete materials, such as strips of paper and algebra tiles, or semi-concrete materials such as the number line. They develop an understanding of numerical operations when they use them regularly to do mental or written computation or computations with a calculator. An understanding of operations is also acquired in a variety of contexts. For example, addition and subtraction can be used in situations that involve uniting, comparing or transforming. Multiplication can be used in cases involving comparison, combination or rectangular arrangement, and division, in situations that involve sharing or capacity (finding the number of times x goes into y).

Concepts	Processes
Understanding Proportionality	***Working With a Proportional Situation***
– Ratio and rate • Ratios and equivalent rates • Unit rate	– Comparison of ratios and of rates
– Proportion • Equality of ratios and rates • Ratio and proportionality coefficient	– Recognizing a proportional situation by referring to the context, a table of values or a graph
– Direct or inverse variation	– Solving a proportional situation
	– Finding ordered pairs in a Cartesian plane [abscissa (*x*-coordinate) and ordinate (*y*-coordinate) of a point]

Learning Process

The development of proportional reasoning is essential, and it has many applications both within and outside mathematics. For example, students use percentages (calculating a certain percentage of a number and the value corresponding to 100 per cent) in situations relating to consumption, probability and statistics. In working with graphs for example, they make scale drawings and construct circle graphs. They look for unknown values in algebraic or geometric situations (e.g. similarity transformations, arc lengths, sector areas, unit conversion).

An understanding of proportions can be developed when students interpret ratios or rates in various situations, compare them qualitatively or quantitatively (e.g. "*a* is darker than *b*," "*c* is less concentrated than *d*") and describe the effect of changing a term, a ratio or a rate. Once students are able to recognize a proportional situation, they can express it as a proportion. They then solve it by using multiplicative strategies that they will have developed (e.g. unit-rate method, factor of change, ratio or proportionality coefficient, additive or mixed procedure). A minimum of three ordered pairs is required to analyze a proportional situation using a table of values.

Examples:

Quantity of product A	2	4	6	10
Quantity of product B	6	12	18	?

Unit-rate method: If for 1 unit of product A, we have 3 units of product B (12 ÷ 4);
then for 10 units of product A, we will have (10 × 3) units of product B.

Factor of change: The factor making it possible for 4 to be increased to 10 is 2.5; we apply this factor to 12.

Proportionality coefficient: The factor making it possible for 4 to be increased to 12 is 3; we apply this factor to 10.

Additive procedure: Since 4:12 = 6:18, then $\frac{4}{12} = \frac{6}{18} = \frac{4+6}{12+18} = \frac{10}{30}$

Algebra

Here algebra behaves, not as a structure among many others, but as a mathematical instrument . . . used in the study of many types of problems . . . .
Seymour Papert

Through their various mathematical activities in elementary school, students were introduced to prerequisites for algebra (e.g. finding unknown terms using properties of operations and relationships between these operations, developing an understanding of equality and equivalence relationships, following the order of operations and looking for patterns in different situations).

In Secondary Cycle One, they develop and master the following concepts and processes:

Concepts

Understanding Algebraic Expressions

– Algebraic expression
  - Variable
  - Coefficient
  - Degree
  - Term, like terms

– Equality, equation and unknown

– First-degree equation with one unknown expressed in the form $ax + b = cx + d$

Processes

– Constructing an algebraic expression

– Recognizing and finding equivalent algebraic expressions

– Numerical evaluation of an algebraic expression

– Manipulating algebraic expressions
  - Addition and subtraction
  - Multiplication and division by a constant
  - Multiplication of first-degree monomials

– Solving first-degree equations with one unknown
  - Validation of the solution by substitution

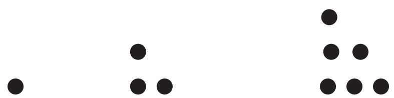
– Overall representation of a situation by means of a graph

Note
The coefficients and the constant terms in algebraic expressions are numbers written in decimal or fractional notation. The type of notation used depends on the situation. For example, when it comes to numbers written in fractional notation, those with a periodic decimal expansion (e.g. $\frac{1}{3}, \frac{2}{7}, \ldots$) and those that can be simplified should not be converted into decimal notation.

Learning Process

In developing their algebraic thinking skills, the students observe patterns related to various situations and represented in different ways (e.g. drawings, tables of values, graphs). Using sequences of numbers is an effective way of introducing the idea of variable, dependence between variables and generalization by means of a rule. For example, polygonal numbers or different geometric situations can be used to generalize by means of one or more equivalent rules:

Examples:

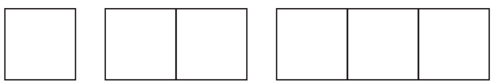


1 3 6

$$\frac{n \times (n+1)}{2}$$

The expression corresponds to the number of dots.

Note: In this example, students use the concept of area and do not have to multiply out this expression.



$1 + n \times 3$

$4 + (n-1) \times 3$

$n \times 4 - (n-1) \times 1$

The expressions correspond to the number of segments.

Translating a word problem into one or more algebraic expressions or equations is one of the tasks involved in solving problems. To become proficient at this, the students must work with a wide variety of situations. Conversely, they will be able to appreciate all the related fine points by translating algebraic expressions into verbal statements or equations into word problems. To enhance their comprehension, they can also use a drawing, a table of values or a graph to represent a situational problem. They are also able to examine and interpret graphic representations of real-world situations.

In the case of algebraic manipulations, students will, if necessary, use drawings or rectangular arrangements when multiplying monomials, for example. They learn to make intradisciplinary and interdisciplinary connections and to transfer knowledge by manipulating algebraic expressions in different situations (e.g. solving a proportion, calculating perimeters or areas, using formulas in a spreadsheet program). These manipulations are used in substituting numerical values and solving equations.

When students substitute numerical values in an algebraic expression to calculate a value, or in an equation to validate their solution, they are applying the properties of arithmetic operations. Moreover, when solving an equation, they must choose the most appropriate method: trial and error, drawings, arithmetic methods (inverse or equivalent operations), algebraic methods (balancing equations, hidden terms).

Below are examples of conjectures that can be used to get students to reason in an arithmetic and algebraic context. Students must justify the steps in their reasoning when concluding that conjectures are true, or produce a counterexample when concluding that they are false.

- *The sum of two consecutive natural numbers is an odd number.*
- *The sum of a sequence of consecutive odd numbers, beginning with one, is a square number.*
- *The sum of two consecutive odd numbers is divisible by 4.*
- *A square number is the sum of two successive triangular numbers.*
- *Given three consecutive numbers, the difference between the square of the second number and the product of the first and third numbers is 1.*
- *The product of two strictly positive numbers is greater than or equal to each of the two numbers.*
- *If an integer is an even number, then it ends with the digit 2.*
- *If an integer ends with the digit 2, then it is an even number.*

Cultural References

Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.
David Hilbert

Mathematics education should include an opportunity for students to learn about how arithmetic and algebra are used in other subject areas (e.g. social sciences, science and technology, the arts). It should also give students the chance to observe the characteristics, advantages and disadvantages of different numeration systems so that they can recognize the numeration system used in their daily life and appreciate its importance. Mathematics education should also introduce students to different types of numbers, such as polygonal and prime numbers, as well as some of their applications (e.g. cryptography). The teacher could present some remarkable sequences of numbers (e.g. the Fibonacci sequence as well as Pascal's triangle and their different applications); suggest situational problems involving arithmetic and algebra that are taken from ancient documents such as the *Rhind papyrus*; offer information on how the use of notation and symbols, computational processes and methods for solving equations has evolved over time; and stimulate discussions on the power and limitations of computational tools (Pascal's adding machine, calculators).

Probability

Life is a school of probability.
Walter Bagehot

In elementary school, students conducted experiments related to the concept of chance. They made qualitative predictions about outcomes by becoming familiar with the following concepts: the certainty, possibility and impossibility of an event and the probability that an event will occur (more likely, just as likely and less likely). They counted the outcomes of a random experiment using tables and tree diagrams, and compared actual outcomes with known theoretical probabilities.

In Secondary Cycle One, they develop and master the following concepts and processes:

Concepts

Random Experiment

- Random experiment
  - Random experiments involving one or more steps (with or without replacement, with or without order)
  - Outcome of a random experiment
  - Sample space
- Event
  - Certain, probable and impossible events
  - Simple, complementary, compatible, incompatible, dependent and independent events
- Theoretical probability and experimental probability

Processes

Processing Data From Random Experiments

- Enumerating possibilities using different types of representations: tree diagram, network, table, etc.
- Calculating the probability of an event

Note
In developing their probabilistic thinking skills, students learn how to use the language of sets, which is considered to be a comprehension and communication tool.

Learning Process

A variety of dynamic learning activities can be used to study probability. In fact, visual information, in the form of experiments, real-world situations, games, diagrams, graphs and sketches, makes it easier to learn about random phenomena and to understand them. Repeating an experiment makes it possible to assimilate certain concepts related to phenomena involving chance. Often, numerous simulations are required before students are able to deal with events that are not equiprobable, to appreciate the significance of certain statements or to detect possible unfairness in the rules of a game, a betting scheme or the interpretation of survey results.

Students develop their probabilistic thinking skills through experimentation. They become interested in verifying their predictions, ask themselves a certain number of questions during simulation activities and discover relationships between the facts they have deemed relevant. The variety of activities allows them to have discussions, adjust their ideas and devise their own models. They develop their critical sense by analyzing and interpreting the resulting probabilities with a view to making decisions or predictions.

Students illustrate and count the different possible outcomes of a random experiment by using tree diagrams, networks or tables. These different representations allow them to deduce the appropriate counting principle in cases where there are too many possible outcomes. In addition, tree diagrams help them to illustrate the probabilities associated with random experiments and to calculate the probability of different events.

Statistics

[Statistics are] the only tools by which an opening can be cut through the formidable thicket of difficulties that bars the path of those who pursue the Science of Man.
Sir Francis Galton

In elementary school, students conducted surveys (they learned how to formulate questions, gather data and organize it using tables). They also interpreted and displayed data using bar graphs, pictographs and broken-line graphs. They interpreted circle graphs and calculated the arithmetic mean of a distribution.

In Secondary Cycle One, they develop and master the following concepts and processes:

Concepts

Statistical Reports

- Population, sample
  - Sample survey, poll, census
  - Representative sample
  - Sampling methods: simple random, systematic
  - Sources of bias
- Data
  - Qualitative variable
  - Discrete or continuous quantitative variable
- Table: characteristics, frequencies
- Reading graphs: bar graphs, broken-line graphs, circle graphs
- Arithmetic mean
- Range

Processes

Processing Data From Statistical Reports

- Conducting a survey or a census
  - Determining the population or the sample
  - Gathering data
- Organizing and choosing certain tools to present data:
  - Constructing tables
  - Constructing graphs: bar graphs, broken-line graphs, circle graphs
  - Highlighting some of the information that can be derived from a table or a graph (e.g. minimum value, maximum value, range, mean)

Learning Process

Statistics helps students develop their critical judgment. To be able to draw conclusions or make informed decisions based on the results of a study or research findings, students must know all the steps involved in conducting a survey. They can learn this by applying each of these steps to a problem they have isolated and that relates to mathematical situations or situations involving other subject areas. They devise a questionnaire and choose a representative sample of the population being studied. They gather data, organize it using a table, display it in graph form and derive information that will allow them to interpret and analyze the results. They choose the graph(s) that provide an appropriate illustration of the situation and compare distributions, if necessary.

Cultural References

There are a wide variety of situations that involve recognizing the concept of chance, interpreting probabilities or understanding statistics. Mathematical learning activities can make students aware of the origin and evolution of random experiments, probability calculations and statistics; stimulate their interest in the mathematicians who contributed to developments in these areas; and teach them to critically analyze games of chance. These learning activities can also illustrate how our relationship to events involving statistics and probability has evolved over time.

GEOMETRY

Geometry is a skill of the eyes and the hands as well as of the mind.
Jean Pedersen

In elementary school, students located numbers on an axis and in a Cartesian plane. They constructed and compared different solids (prisms, pyramids, spheres, cylinders and cones), focusing on prisms and pyramids. They recognized the nets of convex polyhedrons and tested Euler's theorem. They described and classified quadrilaterals and triangles. They are familiar with the features of a circle (radius, diameter, circumference, central angle). They observed and produced frieze patterns and tessellations by means of reflections, rotations and translations. Lastly, they estimated and determined different measurements: lengths, angles, surface areas, volumes, capacities, masses, time and temperature.

In Secondary Cycle One, they develop and master the following concepts and processes:

Concepts

Geometric[12] Figures and Spatial Sense

- Plane figures
  - Triangles, quadrilaterals and regular convex polygons
    - Main segments and lines: bisector, perpendicular bisector, median, altitude
    - Base, height
  - Circle and sector
    - Radius, diameter, chord, arc
    - Central angle
  - Measurement
    - Degree: angle and arc
    - Length
    - Perimeter, circumference
    - Area, lateral area, total area
    - Choice of unit of measure for lengths or areas
    - Relationships between SI[13] units of length
    - Relationships between SI units of area
- Angles
  - Complementary, supplementary
  - Formed by two intersecting lines: vertically opposite, adjacent
  - Formed by a transversal intersecting two other lines: alternate interior, alternate exterior, corresponding

Processes

- Geometric constructions
- Geometric transformations
  - Translation, rotation, reflection
  - Dilatation with a positive scale factor
- Finding unknown measurements
  - Angles
    - Unknown measurement in different situations
  - Lengths
    - Perimeter of a plane figure
    - Circumference of a circle and arc length
    - Perimeter of a figure resulting from a similarity transformation
    - Segments resulting from an isometry or a similarity transformation
    - Unknown measure of a segment in a plane figure
  - Areas
    - Area of polygons that can be split into triangles and quadrilaterals
    - Area of circles and sectors
    - Area of figures that can be split into circles, triangles or quadrilaterals
    - Lateral or total area of right prisms, right cylinders, and right pyramids
    - Lateral or total area of solids that can be split into right prisms, right cylinders or right pyramids

12. In a geometric space of a given dimension (0, 1, 2, or 3), a geometric figure is a set of points representing a geometric object such as a point, line, curve, polygon or polyhedron.
13. International system of Units.

Concepts (*cont.*)

– Solids[14]
  - Right prisms, right pyramids and right cylinders
  - Possible nets of a solid
  - Decomposable solids

– Congruent and similar figures

Processes (*cont.*)

Note

The processes related to geometric transformations and constructions are used to build concepts and identify invariants and properties that can be applied in different situations and for the development of the students' spatial sense. These transformations and constructions can be done using appropriate geometry sets or software in the Euclidian plane. As a result, geometric transformations in the Cartesian plane are not covered in Cycle One.

When determining unknown measurements, students are sometimes required to transfer learning to more complex problems (i.e. those that involve breaking down a problem into subproblems). Calculating the area of decomposable figures is an example of this type of problem. In this way, the students learn how to handle a multi-step problem. They also use the net of a solid. In addition, they use known relations and properties. They apply their arithmetic and algebraic processes as well as proportional reasoning.

14. In this program, students improve the spatial sense they began developing in elementary school. In this regard, refer to the information in the right-hand column on this page and to the learning processes described in the left-hand column on the next page.

Learning Process: Concepts

The statements listed at the end of this section are examples of principles that can be used to get students to reason in a geometric context. Although the properties studied do not necessarily have to be proven, they should represent conclusions that the students will have to draw during exploration activities that require them to use their spatial sense and their knowledge of the properties of geometric transformations, among other things. These statements help them to justify their procedure when solving a situational problem or using mathematical reasoning. When the students are introduced to deductive reasoning, they learn how they can deduce properties using rigorous reasoning based on previously established definitions or properties. Statements 17, 19, 24 and 25 on page 219 can be used to this end.

The use of plane transformations should be regarded as a dynamic way of building geometric concepts and deriving properties and relationships from these concepts, which can then be applied to other situations. By carrying out the steps involved in a construction, the students learn about the fundamental concepts of parallelism, perpendicularity and angles. The numerous observations they can make about a construction also allow them to explore properties of geometric transformations. For example, translations, reflections and rotations are used to introduce the idea of an isometry, and a dilatation with a positive scale factor is used to introduce the idea of a similarity transformation. "Paper-and-pencil" constructions and the use of concrete materials or dynamic geometry software are also ways of building geometric concepts.

To develop their spatial sense in three dimensions, which requires a certain amount of time, the students draw solids freehand. They identify solids by means of their nets or their representations in the plane. They recognize plane figures obtained by the intersection of a solid with a plane.

Learning Process: Processes

The formulas required for measurement are constructed by students through activities that involve "paper-and-pencil" constructions, the use of appropriate software and the manipulation of algebraic expressions, among other things.

In developing their understanding of measurement, the students build the concepts of perimeter and area by comparing perimeters and areas in different situations. They can also make conjectures about the effect of changing a parameter in a formula, for example: "What happens to the perimeter of a rectangle if its dimensions are doubled? What happens to the area of a circle if its radius is doubled? What happens to the area of a rectangle if the length of its base is doubled, tripled or halved?"

In order to determine an unknown measure and justify the steps in their procedure, the students will rely on definitions and properties rather than on measurement. They apply concepts and processes related to arithmetic, algebra and proportions.

The value of geometry lies in the fact that it is used to teach other mathematical concepts as well as concepts related to other subject areas. For example, students use geometric concepts to represent numbers, operations and algebraic expressions. The concepts of similarity and proportionality are used in different graphic representations. In addition, geometric contexts, which involve the concept of area, make it possible to create situations requiring the calculation of probabilities.

Cultural References

Students are required to use their geometric thinking skills and spatial sense in their everyday activities, in different contexts relating to mathematics or other subject areas (e.g. the arts, science and technology), or in different social situations to meet various needs (e.g. getting their bearings, reading a map, evaluating a distance, playing electronic games). They are given the opportunity to learn about mathematicians who shaped the history of geometry and measurement, such as Euclid or Thales. They also study the history of the calculation of the value π, a number that has always fascinated people. They solve measurement problems that many mathematicians have examined throughout history, such as calculating the circumference of the earth (Eratosthenes), the radius of the earth, the distance between the earth and the moon, and the height of a pyramid. Certain measuring instruments have remained virtually unchanged through the ages, and others have been perfected; the students discover them as well as the use of different units of measure.

PRINCIPLES OF EUCLIDIAN GEOMETRY

1. In any isosceles triangle, the angles opposite the congruent sides are congruent.
2. The axis of symmetry of an isosceles triangle coincides with a median, a perpendicular bisector, an angle bisector and an altitude of the triangle.
3. The opposite sides of a parallelogram are congruent.
4. The diagonals of a parallelogram bisect each other.
5. The opposite angles of a parallelogram are congruent.
6. The diagonals of a rectangle are congruent.
7. The diagonals of a rhombus are perpendicular to each other.
8. If two lines are parallel to a third line, then they are parallel to each other.
9. If two coplanar lines are perpendicular to a third line, then they are parallel to each other.
10. If two lines are parallel, any line perpendicular to one of these lines is perpendicular to the other.
11. Three non-collinear points determine one and only one circle.
12. All the perpendicular bisectors of the chords of a circle meet at the centre of the circle.
13. All the diameters of a circle are congruent.
14. In a circle, the measure of a radius is equal to half the measure of the diameter.
15. The ratio of the circumference of a circle to its diameter is a constant known as π.
16. Adjacent angles whose external sides are in a straight line are supplementary.
17. Vertically opposite angles are congruent.
18. In a circle, the degree measure of the central angle is equal to the degree measure of its intercepted arc.
19. If a line intersects two parallel lines, then the alternate interior angles are congruent, the alternate exterior angles are congruent and the corresponding angles are congruent.
20. In the case of a line that intersects two lines, if two corresponding (or alternate interior or alternate exterior) angles are congruent, then the two lines are parallel.
21. If a line intersects two parallel lines, then the pairs of interior angles on the same side of the transversal are supplementary.
22. In a circle, the ratio of the measures of two central angles is equal to the ratio of the measures of their intercepted arcs.
23. In a circle, the ratio of the areas of two sectors is equal to the ratio of the measures of their central angles.
24. The sum of the measures of the interior angles of a triangle is 180°.
25. The measure of an exterior angle of a triangle is equal to the sum of the measures of the two non-adjacent (or remote) interior angles.
26. The corresponding elements of congruent plane figures or solids are equal in measure.
27. The corresponding angles of similar plane figures or solids are congruent, and the measures of their corresponding sides are proportional.
28. In similar plane figures, the ratio of the areas is equal to the square of the ratio between the lengths of the corresponding sides.

Examples of Strategies Employed in Solving Situational Problems and That Students Can Learn to Use When Exercising Their Competencies

Each problem that I solved became a rule which served afterwards to solve other problems.
René Descartes

Understanding

- Distinguishing between everyday language and mathematical language
- Conceptualizing the situation mentally or in writing
- Defining the task to be performed
- Reformulating the situation in one's own words

Organizing

- Making connections
- Identifying relevant concepts and processes
- Using lists, tables, diagrams, concrete materials and drawings

Solving

- Using a trial-and-error approach
- Working backwards
- Referring to a similar problem that he/she has already solved
- Breaking down a complex problem into subproblems
- Simplifying the problem

Validating

- Verifying one's solutions by means of examples or reasoning
- Using other processes, if necessary
- Looking for counterexamples
- Comparing procedures and final answers with those of a teacher or of classmates

Communicating

- Organizing his/her ideas
- Compare their understanding of everyday words with the meaning these same words have in mathematical language
- Using different types of representation
- Experimenting with different ways of conveying a mathematical message
- Explaining his/her reasoning

Bibliography

General Principles

Abrantes, Paulo, Lurdes Serrazina, and Isolina Oliveira. *A Matemática na Educaçïo Básica*. Lisbon: Departamento da Educaçïo Básica, 1999.

Barberousse, Anouk. "Un dédale conceptuel dans l'empire des probabilités, Dieu joue-t-il aux dés?". *Sciences et avenir*. Paris, hors-série, October-November 2001, no. 128: 16-22.

Burke, Maurice J., and Frances R. Curcio, ed. *Learning Mathematics for a New Century: 2000 Yearbook*. Reston, Virginia: National Council of Teachers of Mathematics, 2000.

Cuoco, Albert A., and Frances R. Curcio, ed. *The Roles of Representation in School Mathematics: 2001 Yearbook*. Reston, Virginia: National Council of Teachers of Mathematics, 2001.

Department for Education and Employment of England. *The National Curriculum for England: Mathematics*. [On-line], 1999, <http://www.nc.uk.net>.

Department of Education, Government of Western Australia. *Mathematics: Student Outcome Statements*. [On-line], 1998, <http://www.eddept.wa.edu.au/outcomes/maths/mathmenu.htm> (September 2000).

House, Peggy A., and Arthur F. Coxford, ed. *Connecting Mathematics Across the Curriculum: 1995 Yearbook*. Reston, Virginia: National Council of Teachers of Mathematics, 1995.

Mason, John, Leone Burton, and Kay Stacey. *Thinking Mathematically*. Reading, Massachusetts: Addison-Wesley, 1982.

National Council of Teachers of Mathematics. *Principles and Standards for School Mathematics*. Reston, Virginia: National Council of Teachers of Mathematics, 2000.

Paulos, John Allen. *Innumeracy, Mathematical Illiteracy and Its Consequences*. New York: Vintage Books Edition, 1990.

Portugal. Departamento da Educaçïo Básica do Portugal. *Currículo Nacional: Competências Essenciais – Matemática*, 2001, 57-71.

Québec. Ministère de l'Éducation. *Curriculum Guide, Elementary School, Mathematics, Booklet K, Problem Solving, General Orientations*. (Code: 16-2300-11A). Québec: Gouvernement du Québec, 1989.

Stiff, Lee V., and Frances R. Curcio, ed. *Developing Mathematical Reasoning in Grades K-12, 1999 Yearbook*. Reston, Virginia: National Council of Teachers of Mathematics, 1999.

Walter, Anne. "Quelle géométrie pour l'enseignement en Collège?" *Petit x*, 2000-2001, no. 54: 31-49.

Educational Theory

Arsac, Gilbert, et al. *Initiation au raisonnement déductif au collège*. Lyon: Presses Universitaires de Lyon, 1992.

Arsac, Gilbert. "Un cadre d'étude du raisonnement mathématique." In Grenier D., ed. *Séminaire didactique et technologies cognitives en mathématiques.* Grenoble: IMAG, 1996.

Barbin, Evelyne. "Histoire et enseignement des mathématiques : Pourquoi? Comment?" *Bulletin AMQ*, March 1997, Vol. XXXVII, no. 1: 20-25.

Becker, Jerry P., and Shigeru Shimada, eds. *The Open-Ended Approach, A New Proposal for Teaching Mathematics*. Reston, Virginia: National Council of Teachers of Mathematics, 1997.

Descaves, Alain. *Comprendre des énoncés, résoudre des problèmes*. Collection Pédagogies pour demain. Didactiques 1er degré. Paris: Hachette Éducation, 1992.

Duperret, Jean-Claude. «Le geste géométrique, ou l'acte de démontrer." *Repères, Revue des IREM*, April 2001, no. 43: 83-116.

Duval, Raymond. "Pour une approche cognitive de l'argumentation." *Annales de didactique et de sciences cognitives 3*.Strasbourg: IREM de Strasbourg, 1990, 195-221.

Edwards, Edgar L. Jr. *Algebra for Everyone*. Reston, Virginia: National Council of Teachers of Mathematics, 1990.

Friel, Susan, Sid Rachlin, and Dot Doyle. *Navigating Through Algebra in Grades 6-8*. Navigations Series. Reston, Virginia: National Council of Teachers of Mathematics, 2001.

Gros, Daniel. "Une enquête statistique au service de la proportionnalité." *Repères, Revue des IREM*. July 2001, no. 44: 69-81.

Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell. *Adding It Up, Helping Children Learn Mathematics*. Washington: National Academy Press, 2001.

Litwiller, Bonnie, and George Bright, ed. *Making Sense of Fractions, Ratios and Proportions: 2002 Yearbook and Classroom Activities*. Reston, Virginia: National Council of Teachers of Mathematics, 2002.

Owens, Douglas T., ed. *Research Ideas for the Classroom: Middle Grades Mathematics*. New York: Macmillan Library Reference, 1993.

Pugalee, David K., et al. *Navigating Through Geometry in Grades 6-8*. Navigations Series. Reston, Virginia: National Council of Teachers of Mathematics, 2001.

René de Cotret, Sophie. *Étude de l'influence des variables : indice de proportionnalité du thème et nombre de couples de données sur la reconnaissance. Le traitement et la compréhension de problèmes de proportionnalité chez les élèves de 13-14 ans*. Doctoral Dissertation, Grenoble: Université Joseph Fourier, 1991.

Richard, Philippe R. *Modélisation du comportement en situation de validation: Diagnostic sur les stratégies de preuve et sur leur raisonnement employés en géométrie par des élèves de niveau secondaire*. Doctoral Dissertation, Barcelona: Universitat autònoma de Barcelona, 2000.

Rouche, Nicolas. *Pourquoi ont-ils inventé les fractions?* Collection L'esprit des sciences. Paris: Ellipses, 1998.

Van De Walle, John A. *Elementary School Mathematics: Teaching Developmentally*. 2nd edition. New York: Longman, 1994.

Reference Books

Baruk, Stella. *Dictionnaire de mathématiques élémentaires*. Paris: Seuil, 1992.

Bouvier, Alain, and Michel George. *Dictionnaire des mathématiques*. François Le Lyonnais, ed. Paris: PUF, 1983.

Burton, David M. *The History of Mathematics: An Introduction*. Dubuque: WCB, 1988.

Mankiewicz, Richard. *The Story of Mathematics*. Princeton, N.J.: Princeton University Press, 2000.

Québec. Ministère de l'Éducation. *Information Document: Graphs, Notation and Symbols Used in Secondary School Mathematics*. (Code:16-3306A). Québec: Gouvernement du Québec, 1997.