Clarifications regarding the types of reasoning to be used in Secondary School Mathematics

Examples of approaches illustrating the different types of reasoning

Québec

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Cycle One

Confirm or refute the following statement: When two opposite numbers are added to a statistical distribution, the mean does not change.

➲ Example of an approach that involves refuting a statement by finding a counterexample

① I'll try to validate the statement by using the following statistical distribution: $2, 3$ and 4.

② I'll calculate the mean of the distribution.

The mean is equal to:
the sum of the data values ÷ the total number of data values $= (2+3+4) \div 3 = 9 \div 3 = 3$.

The mean of the initial distribution is 3.

③ Now I'll check whether the mean changes when the following opposite numbers are added: 5 and -5.

The mean of the new distribution is equal to $\big(2 + 3 + 4 + 5 + (-5)\big) \div 5 = 9 \div 5 = 1.8$.

④ My conclusion: the statement is false because there is a counterexample showing that when two opposite numbers are added to the initial distribution, the mean turns out to be different ($3 \neq 1.8$).

Enrichment: When students have answered the question correctly, they can be asked whether they would obtain the same result if all the numbers in the initial distribution were negative.

➲ Example of an approach that involves deductive reasoning

① I identify the unknowns and the variable.

n : Number of data values in the initial distribution, where $n \in \mathbb{N}^*$
a : Sum of the data values in the initial distribution, where $a \in \mathbb{R}$
x : A given number, where $x \in \mathbb{R}$
$-x$: Opposite of the given number

② I calculate the mean of the initial distribution.

The mean is equal to $\dfrac{\text{the sum of the data values}}{\text{the total number of data values}} = \dfrac{a}{n}$

③ If I add two opposite numbers to the distribution, the mean is then equal to:

The sum of two opposite numbers is equal to 0.

$$\frac{x + (-x) + a}{n+2} = \frac{a}{n+2}$$

④ I compare the two means.

$$\frac{a}{n+2} \neq \frac{a}{n}$$

Therefore, the statement is false. When two opposite numbers are added to a statistical distribution, the mean changes.

A student's deductive reasoning does not have to be algebraic; it can be presented in words only.

Enrichment: When students have answered the question correctly, they can be asked to find the specific case where the statement is true.

➲ Example of an approach that involves analogical reasoning

- If I add two opposite numbers to a sum, it's like adding nothing at all because the sum of two opposite numbers is always equal to zero.

- So, if two opposite numbers are added, the sum of the data values in the distribution remains the same.

- To calculate the mean, I divide the sum of the data values by the number of data values. In this case, I have to divide the same sum by a greater value (two more data values). For example, if I cut a cake into 5 equal parts, I will have bigger pieces than if I cut it into 7 equal parts. The resulting mean will therefore be less than, and not equal to, the mean of the initial distribution.

The statement is therefore false. When two opposite numbers are added to a statistical distribution, the mean changes.

We have not provided an example of inductive reasoning for this question, since the assertion is false. Students who use this type of reasoning should end up with a counterexample that refutes the initial claim.

Is the following statement true or false? The sum of two negative numbers is always negative.

➲ Example of an approach that involves analogical reasoning

To explain negative numbers more clearly, let's use an example involving money.

I had no money yesterday, so I asked my friend Louis for a loan. I now have a debt (a value corresponding to a negative number). I still haven't paid my friend back, and today I wanted to buy a snack. Louis said he'd lend me some more money, so I have another debt (a second value corresponding to a negative number).

If I add up the two debts (the sum of two negative numbers), I end up with one big debt (but the result is still negative).

The statement is therefore true. The sum of two negative numbers is always negative because if I had a debt yesterday and take on another debt today, I will then have a bigger debt, which is still an amount of money less than $0.

> In analogical reasoning, explanations are not based on only one example. They are general in nature and cover as many cases as possible.

Secondary III

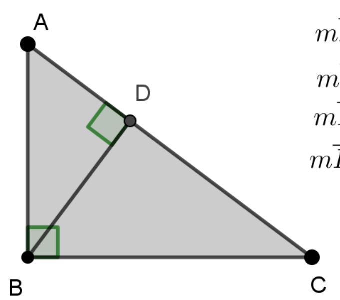
In a right triangle, an altitude is drawn from the vertex of the right angle. What is the relationship between the lengths of the legs, the length of the hypotenuse and the length of the altitude drawn? Explain.

➲ Example of an approach that involves inductive reasoning

Using Pythagorean triples and dynamic geometry software, I will construct two different right triangles and determine the length of the altitudes drawn from the vertex of the right angle.

I will avoid using similar triangles to ensure that I have a variety of examples.

Example 1



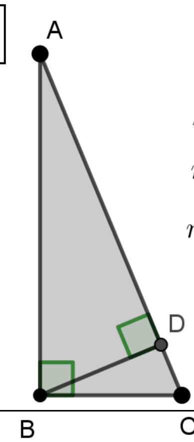
$m\overline{AB} = 3\text{cm}$

$m\overline{AC} = 5\text{cm}$

$m\overline{BC} = 4\text{cm}$

$m\overline{BD} = 2.4\text{cm}$

Example 2



$m\overline{AC} = 13\text{cm}$

$m\overline{AB} = 12\text{cm}$

$m\overline{BC} = 5\text{cm}$

$m\overline{BD} = \frac{60}{13}\text{cm}$

Note that
$$3 \times 4 = 2.4 \times 5$$
$$12 = 12$$

and that
$$12 \times 5 = \frac{60}{13} \times 13$$
$$60 = 60$$

In my two examples, ***the product of the lengths of the legs is equal to the product of the length of the hypotenuse and the length of the altitude drawn from the vertex of the right angle***.

This relationship seems correct because, to calculate the area of a triangle, you can use any of its three sides as the base along with the corresponding height.

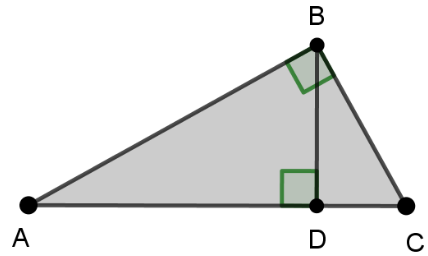
Calculation of the area of a right triangle if the base is

a leg:	the hypotenuse:
Area of triangle ABC = $\frac{m\,\overline{BC} \cdot m\,\overline{AB}}{2}$	Area of triangle ABC = $\frac{m\,\overline{AC} \cdot m\,\overline{BD}}{2}$

Since the area remains the same regardless of the side used as the base, the result is $\frac{m\,\overline{BC} \cdot m\,\overline{AB}}{2} = \frac{m\,\overline{AC} \cdot m\,\overline{BD}}{2}$, meaning that $\overline{BC} \cdot m\,\overline{AB} = m\,\overline{AC} \cdot m\,\overline{BD}$, which is the relationship stated above.

➲ Example of an approach that involves deductive reasoning

Consider right scalene triangle ABC, which is right-angled at B, with an altitude ($\overline{BD}$) drawn from the vertex of the right angle.



I notice that the altitude forms two other right triangles: triangles ADB and BDC. In addition,

1. $m\angle BAC = m\angle BAD$	1. Angle common to triangles ABC and ADB
2. $m\angle ABC = m\angle ADB = 90°$	2. By definition of a right triangle and an altitude
3. a) $m\angle ACB = 90° - m\angle BAC$ b) $m\angle DBA = 90° - m\angle BAD$	3. The acute angles of a right triangle are complementary.
4. $m\angle ACB = m\angle DBA$	4. I subtracted the same measure from 90°, since $m\angle BAC = m\angle BAD$ (proved in step 1)

Triangles ADB and ABC are similar because their corresponding angles are congruent. The lengths of the corresponding sides enable me to find the similarity ratio of the two triangles.

Similarity ratio: $\frac{m\,\overline{AB}}{m\,\overline{AD}} = \frac{m\,\overline{BC}}{m\,\overline{DB}} = \frac{m\,\overline{AC}}{m\,\overline{AB}}$

I can use this proportion to examine the relationship in question because its components represent the lengths of the legs, the length of the hypotenuse and the length of the altitude drawn.

So, the ratio between the length of the shortest leg and the length of the altitude drawn from the vertex of the right angle is equal to the ratio between the length of the hypotenuse and the length of the longest leg.

This task can also be assigned to Cycle One students if dynamic geometry software is used to construct the triangle, or to Secondary IV students who are not yet familiar with metric relations in right triangles.

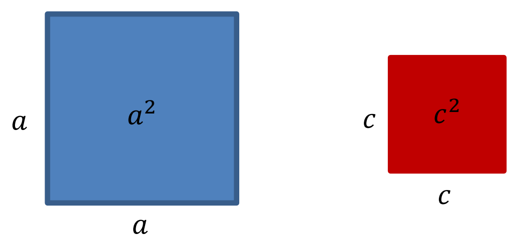
Secondary IV

Show that the following expressions are equivalent:

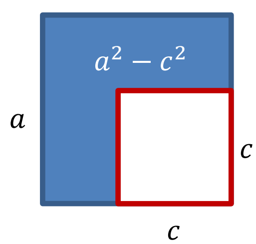
$$(a^2 - c^2) \text{ and } (a + c)(a - c)$$

➲ Example of an approach that involves analogical reasoning

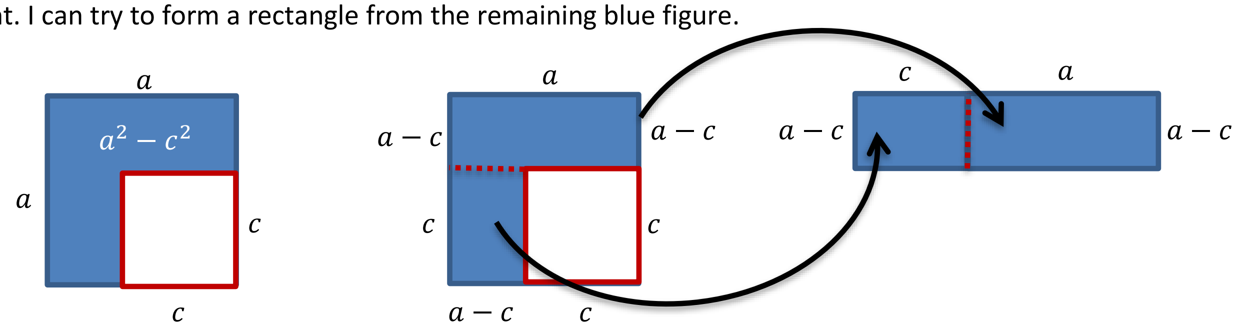
The binomial $(a^2 - c^2)$ corresponds to a difference of two squared terms. I can represent this expression by using algebra tiles, where a^2 represents a square with each side measuring a units, and c^2 represents a square with each side measuring c units.



I can represent the difference $(a^2 - c^2)$ by subtracting the area of the red square (c^2) from the area of the blue square (a^2). The resulting geometric shape is the remaining blue figure below.



The expression $(a + c)(a - c)$ is the product of two factors. It must also represent the area of the remaining blue figure. I know that I can calculate the area of a rectangle by multiplying the measure of the base by the height. I can try to form a rectangle from the remaining blue figure.



The dimensions of the resulting rectangle are $(a + c)$ and $(a - c)$. We can calculate the area of the remaining blue figure by multiplying the measure of the base by the height $(a + c)(a - c)$ or by subtracting the area of one square from the area of another square, namely $(a^2 - c^2)$. Therefore, $(a^2 - c^2) = (a + c)(a - c)$.

➲ Example of an approach that involves deductive reasoning in the form of an algebraic proof

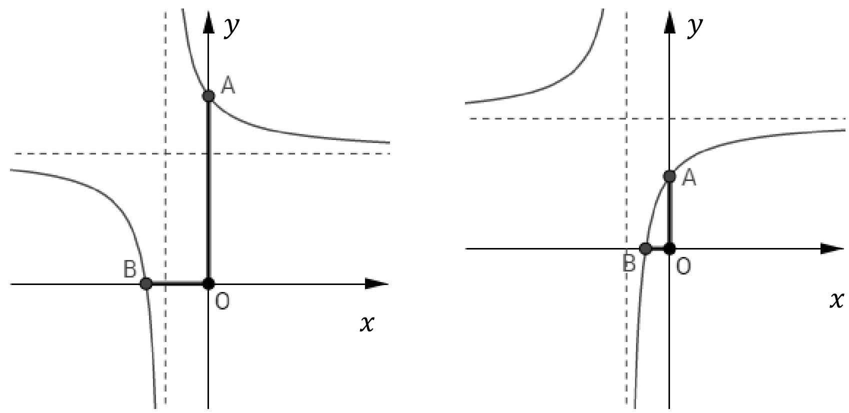
By expanding the expression $(a+c)(a-c)$, I obtain:

$(a+c)(a-c)$	$=$	$a(a-c)+c(a-c)$	Double distributive property
	$=$	$a^2-ac+ac-c^2$	Distributive property of multiplication over addition
	$=$	a^2-c^2	Simplification by adding similar terms

Through a series of algebraic manipulations based on the properties associated with the multiplication and addition of terms, I have shown that $(a^2-c^2)=(a+c)(a-c)$.

Secondary V

Below are two possible graphical representations of a rational function. Make a conjecture about the relationship between the ratio $\frac{m\,\overline{OA}}{m\,\overline{OB}}$ and parameters *h* and *k* of the rule of the function.



➲ Example of an approach that involves inductive reasoning

Consider the rule of the rational function in standard form $f(x) = a\left(\frac{1}{b(x-h)}\right) + k$

1. I will consider a few examples where h and k are assigned different values (positive, negative, rational and irrational values).

Example ① Let $a = 1$, $b = 1$, $h = 4$ and $k = 2$

Therefore, $f(x) = \frac{1}{(x-4)} + 2$

y-intercept (Point A)

$$f(0) = \frac{1}{(0-4)} + 2$$

$$f(0) = \frac{7}{4}$$

x-intercept (Point B)

$$0 = \frac{1}{(x-4)} + 2$$

$$-2 = \frac{1}{(x-4)}$$

$$-2(x-4) = 1$$

$$-2x + 8 = 1$$

$$-2x = -7$$

$$x = \frac{7}{2}$$

- $m\,\overline{OA} = \frac{7}{4}$
- $m\,\overline{OB} = \frac{7}{2}$
- $\frac{m\,\overline{OA}}{m\,\overline{OB}} = \frac{7/4}{7/2} = \frac{2}{4}$

The ratio $\frac{m\,\overline{OA}}{m\,\overline{OB}}$ is equal to the ratio $\frac{k}{h}$ $\left(\frac{2}{4}\right)$ in this example. I'll see if this is true in other cases.

Example ② Let $a = 1$, $b = 1$, $h = -5$ and $k = \frac{6}{7}$

Therefore, $f(x) = \frac{1}{(x+5)} + \frac{6}{7}$

y-intercept (Point A)

$$f(0) = \frac{1}{5} + \frac{6}{7}$$

$$f(0) = \frac{37}{35}$$

x-intercept (Point B)

$$0 = \frac{1}{(x+5)} + \frac{6}{7}$$

$$-\frac{6}{7} = \frac{1}{(x+5)}$$

$$-6(x+5) = 7$$

$$-6x - 30 = 7$$

$$-6x = 37$$

$$x = -\frac{37}{6}$$

- $m\,\overline{OA} = \frac{37}{35}$
- $m\,\overline{OB} = -\frac{37}{6}$
- $\frac{m\,\overline{OA}}{m\,\overline{OB}} = \frac{37/35}{-37/6} = -\frac{6}{35}$

I'll see if, in example ②, $\frac{m\,\overline{OA}}{m\,\overline{OB}}$ is equal to the ratio between the values of parameters *k* and *h*.

$\frac{k}{h} = \frac{6/7}{-5} = -\frac{6}{35}$. In this case, it is.

Now I'll see if this is still true even if the value of parameters *a* and *b* is not equal to 1.

Example ③ Let $a = -2$, $b = 9$, $h = \frac{7}{3}$ and $k = \sqrt{2}$

Therefore, $f(x) = \frac{-2}{9(x-7/3)} + \sqrt{2}$

y-intercept (Point A)

$$f(0) = \frac{-2}{9(-7/3)} + \sqrt{2}$$

$$f(0) = \frac{-2}{-21} + \sqrt{2}$$

$$f(0) = \frac{2 + 21\sqrt{2}}{21}$$

x-intercept (Point B)

$$0 = \frac{-2}{9(x-7/3)} + \sqrt{2}$$

$$-\sqrt{2} = \frac{-2}{9(x-7/3)}$$

$$-9\sqrt{2}(x - 7/3) = -2$$

$$-9\sqrt{2}x + 21\sqrt{2} = -2$$

$$-9\sqrt{2}x = -2 - 21\sqrt{2}$$

$$x = \frac{-2 - 21\sqrt{2}}{-9\sqrt{2}}$$

$$x = \frac{21 + \sqrt{2}}{9}$$

- $m\,\overline{OA} = \frac{2+21\sqrt{2}}{21}$
- $m\,\overline{OB} = \frac{21+\sqrt{2}}{9}$
- $\frac{m\,\overline{OA}}{m\,\overline{OB}} = \frac{\frac{2+21\sqrt{2}}{21}}{\frac{21+\sqrt{2}}{9}} = \frac{3\sqrt{2}}{7}$

$$\frac{2 + 21\sqrt{2}}{21} \div \frac{21 + \sqrt{2}}{9} =$$

$$\frac{2 + 21\sqrt{2}}{21} \times \frac{9}{21 + \sqrt{2}} =$$

$$\frac{(6 + 63\sqrt{2}) \cdot (147 - 7\sqrt{2})}{(147 + 7\sqrt{2}) \cdot (147 - 7\sqrt{2})} =$$

$$\frac{3\sqrt{2}}{7}$$

Let's see if my observation is still true. In example ③, the ratio $\frac{k}{h} = \frac{\sqrt{2}}{7/3} = \frac{3\sqrt{2}}{7}$, which corresponds to the ratio $\frac{m\,\overline{OA}}{m\,\overline{OB}}$.

2. For rational functions, I can make the following conjecture: the ratio between the *y*-intercept and the *x*-intercept $\left(\frac{m\,\overline{OA}}{m\,\overline{OB}}\right.$ in the question$\left.\right)$ is equal to the ratio between the values of parameters *k* and *h*. Thus, $\frac{m\,\overline{OA}}{m\,\overline{OB}} = \frac{k}{h}$.

   However, I notice that $m\,\overline{OB}$ and parameter h cannot be equal to zero, because the ratios would be undefined (since division by zero is impossible). I also notice that the function should not pass through the origin ($m\,\overline{OA} = 0$), because that would make the ratio equal to zero. This means that the value of parameter k of the function should also not be equal to zero. The function should therefore have undergone a horizontal translation and a vertical translation.

The first step in tackling this question is to explore different examples. In order to provide a complete line of inductive reasoning, students must refer to the properties of hyperbolas to identify the constraints under which their conjecture is true. The conjecture is still not proven in the work shown above.

➲ Example of an approach that involves deductive reasoning

Consider the rule of the rational function in standard form $f(x) = a\left(\frac{1}{b(x-h)}\right) + k$, where $b \neq 0$.

① If there is no horizontal or vertical translation of the function centred at the origin, points A and B (points of intersection with the axes) do not exist. The ratio $\frac{m\,\overline{OA}}{m\,\overline{OB}}$ is then undefined because the *x*-axis and the *y*-axis are also the horizontal and vertical asymptotes of the function. For the function to pass through each axis, the values of parameters k and h must not be equal to zero. In addition, if the *x*-intercept (Point B) is zero, the ratio will also be undefined (since division by zero is impossible). The following proof is true if the rational function is not centred at the origin of the Cartesian plane.

② To determine $m\overline{OA}$, find the *y*-intercept:

Let $x = 0$ in $f(x) = a\left(\frac{1}{b(x-h)}\right) + k$

$$f(0) = a\left(\frac{1}{b(0-h)}\right) + k$$

$$f(0) = -\frac{a}{bh} + k$$

The *y*-intercept is $-\frac{a}{bh} + k$.

Therefore, $\overline{m\ OA} = -\frac{a}{bh} + k$.

③ To determine $m\overline{OB}$, find the *x*-intercept:

Let $f(x) = 0$ in $f(x) = a\left(\frac{1}{b(x-h)}\right) + k$

$$\begin{aligned} 0 &= a\left(\frac{1}{b(x-h)}\right) + k \\ -k &= \frac{a}{b(x-h)} \\ -kb(x-h) &= a \\ -kbx + kbh &= a \\ kbh &= a + kbx \\ kbh - a &= kbx \\ h - \frac{a}{kb} &= x \end{aligned}$$

The *x*-intercept is $h - \frac{a}{bk}$.

Therefore, $\overline{m\ OB} = h - \frac{a}{bk}$.

④ Calculate the value of the ratio $\frac{m\,\overline{OA}}{m\,\overline{OB}}$:

$$\frac{m\,\overline{OA}}{m\,\overline{OB}} = m\,\overline{OA} \div m\,\overline{OB} = \left(-\frac{a}{bh} + k\right) \div \left(h - \frac{a}{bk}\right) =$$

$$\left(\frac{-a}{bh} + \frac{bhk}{bh}\right) \div \left(\frac{bhk}{bk} - \frac{a}{bk}\right) = \left(\frac{-a + bhk}{bh}\right) \div \left(\frac{bhk - a}{bk}\right) =$$

$$\left(\frac{-a + bhk}{bh}\right) \cdot \left(\frac{bk}{bhk - a}\right) = \frac{-abk + b^2hk^2}{b^2h^2k - abh} = \frac{bk(-a + bhk)}{bh(bhk - a)} = \frac{k}{h}$$

The value of the ratio $\frac{m\,\overline{OA}}{m\,\overline{OB}} = \frac{k}{h}$.

This algebraic proof leads to the following conjecture:

If $h \neq 0$, $k \neq 0$ and the function is not centred at the origin, then the ratio $\frac{m\,\overline{OA}}{m\,\overline{OB}}$ is equal to the ratio between the value of parameter *k* and the value of parameter *h* $\left(\frac{m\,\overline{OA}}{m\,\overline{OB}} = \frac{k}{h}\right)$.